Theorem: Consider the autonomous differential equation

$$x'(t) = f(x)$$

with $f(x)$ and $\frac{\partial}{\partial x} f(x)$ continuous (so local existence and uniqueness theorems hold). Let $f(c) = 0$, i.e. $x(t) \equiv c$ is an equilibrium solution. Suppose c is an *isolated zero* of *f*, i.e. there is an open interval containing c so that c is the only zero of f in that interval. The the stability of the equilibrium solution c can is completely determined by the local phase diagrams:

$sign(f):\ \underline{----}\,\underset{c}{0}\,\underline{+++}\quad \Rightarrow \quad \leftarrow\leftarrow\leftarrow c \rightarrow\rightarrow\rightarrow \quad \Rightarrow$ c is unstable

$sign(f):\ +++0---\quad \Rightarrow \quad \rightarrow\rightarrow\rightarrow c \leftarrow\leftarrow\leftarrow \quad \Rightarrow$ c is asymptotically stable

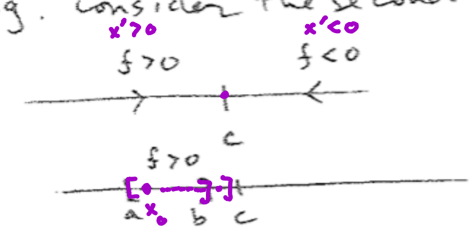
$sign(f):\ +++0+++\quad \Rightarrow \quad \rightarrow\rightarrow\rightarrow c \rightarrow\rightarrow\rightarrow \quad \Rightarrow$ c is unstable (half stable)

$sign(f):\ ----0---\quad \Rightarrow \quad \leftarrow\leftarrow\leftarrow c \leftarrow\leftarrow\leftarrow \quad \Rightarrow$ c is unstable (half stable)

You can actually prove this Theorem with calculus!! (want to try?)

Here's why!

e.g. consider the second case



f cont; $f>0$ on subinterval $[a,b]$

$\Rightarrow f \geq \delta > 0$ on $[a,b]$

$x'(t) = f(x) \geq \delta$ as long as $x \in [a,b]$

(extreme value thm from calculus, f attains its minimum)

$\Rightarrow x'(t) \geq \delta$ as long as $x(t) \in [a,b]$

$\Rightarrow x(t)$ stays in this interval for time interval at most $\frac{b-a}{\delta}$

speed · time = dist

time = $\frac{\text{dist}}{\text{speed}}$.

$\Rightarrow \lim_{t\to\infty} x(t) = c$

(because we can pick the right endpoint b as close to c ($b<c$) as we want, to ensure $x(t)$ eventually gets as close as we want to the value c)

Exercise 3) Use the chain rule to check that if $x(t)$ solves the autonomous DE

$$x'(t) = f(x) : \ x'(\tau) = f(x(\tau))$$

Then $\boxed{X(t) := x(t - a)}$ solves the same DE. What does this say about the geometry of representative solution graphs to autonomous DEs? Have we already noticed this?

Start here Friday: Check: LHS: $X'(t) = x'(t-a)\cdot 1 = f(x(t-a)) =$
$= f(X(t))$

Further application: Doomsday-extinction. With different hypotheses about fertility and mortality rates, one can arrive at a population model which looks like logistic, except the right hand side is the opposite of what it was in that case:

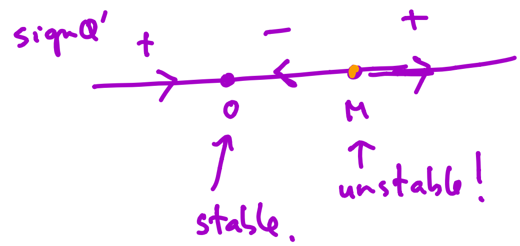
Logistic: $P'(t) = -a\,P^2 + b\,P$ $\quad kP(M-P)$

Doomsday-extinction: $Q'(t) = \underline{a\,Q^2} - \underline{b\,Q}$

For example, suppose that the chances of procreation are proportional to population density (think alligators or crickets), i.e. the fertility rate $\beta = a\,Q(t)$, where $Q(t)$ is the population at time t. Suppose the morbidity rate is constant, $\delta = b$. With these assumptions the birth and death rates are $a\,Q^2$ and $-b\,Q$ .... which yields the DE above. In this case factor the right side:

$$Q'(t) = a\,Q\left(Q - \frac{b}{a}\right) = k\,Q(Q - M). \qquad kQ(Q-M)$$

Exercise 4a) Construct the phase diagram for the general doomsday-extinction model and discuss the stability of the equilbrium solutions.



[$Q_0 > M$ "doomsday" in finite time

$Q_0 < M$ $Q(t) \to 0$ as $t \to \infty$ "extinction"

Exercise 4b) If $P(t)$ solves the logistic differential equation

$$P'(t) = k\,P(M - P)$$

opposites

show that $Q(t) := P(-t)$ solves the doomsday-extinction differential equation

$$Q'(t) = k\,Q(Q - M)\,.$$

Use this to recover a formula for solutions to doomsday-extinction IVPs. What does this say about how representative solution graphs are related, for the logistic and the doomsday-extinction models? Recall, the solution to the logistic IVP is

$$P(t) = \frac{MP_0}{(M - P_0)e^{-Mkt} + P_0}\,.$$

$$\text{IVP}\begin{cases} Q'(t) = k\,Q(Q-M) \\ Q(0) = Q_0 \end{cases}$$

$$Q(t) = \frac{MQ_0}{(M-Q_0)e^{Mkt} + Q_0}$$

$Q_0 > M$: $|M - Q_0| < Q_0$

if $x(t)$ solves
$x' = f(x)$

then $z(t) = x(-t)$ solves
$z' = -f(z)$

Check:

$$\begin{aligned} z'(t) &= x'(-t)\cdot(-1) \\ &= -x'(-t) \\ &= -f(x(-t)) \\ &= -f(z(t))! \end{aligned}$$

Exercise 5: Use your formula from the previous exercise or work the separable DE from scratch, to transcribe the solution to the doomsday-extinction IVP

$$x'(t) = x(x - 1)$$
$$x(0) = 2\,.$$

$M = 1$
$Q_0 = 2$

Does the solution exist for all $t > 0$? (Hint: no, there is a very bad doomsday at $t = \ln 2$.)

$$x(t) = \frac{2}{-e^t + 2}\,.$$

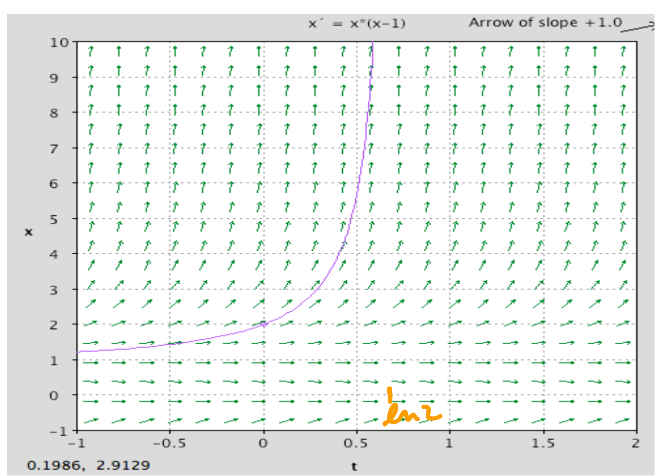
$$x(0) = \frac{2}{-1+2} = 2 \checkmark$$

vertical asymptote

@ $-e^t + 2 = 0$

$2 = e^t$

$\ln 2 = t$



Math 2280-001
Fri Jan 30
2.2 Autonomous differential equations, with applications; 2.3 improved velocity models

Friday: • finish Wed notes
• do §2.2 part of todays

Monday.

- Recall that on Wednesday we discussed the following important concepts:
  - * Autonomous first order DE
  - * equilibrium solutions for autonomous DE's
  - * stability at equilibrium points.

Further application: (related to parts of a "yeast bioreactor" homework problem for next week) harvesting a logistic population...text p.89-91 (or, why do fisheries sometimes seem to die out "suddenly"?)
Consider the DE

logistic — harvesting term.

$$P'(t) = a\,P - b\,P^2 - h\,. \quad = 2P - P^2 - h$$

Notice that the first two terms represent a logistic rate of change, but we are now harvesting the population at a rate of h units per time. For simplicity we'll assume we're harvesting fish per year (or thousands of fish per year etc.) One could model different situations, e.g. constant "effort" harvesting, in which the effect on how fast the population was changing could be $h\,P$ instead of P .

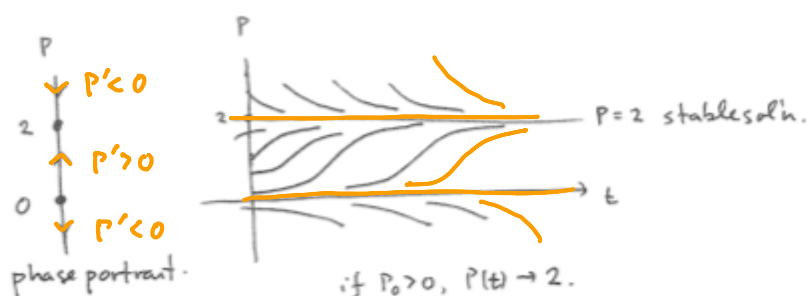
For computational ease we will assume $a = 2$, $b = 1$. (One could actually change units of population and time to reduce to this case.)

for computational simplicity
take $a=2$, $b=1$

Case 0 no harvesting

standard logistic

$$P'(t) = 2P - P^2 = P(2-P)$$



if $P_0>0$, $P(t)\to 2$.
(units could be millions of fish, e.g.)

with harvesting:

$$P'(t) = 2P - P^2 - h\,.$$

factor out −1

$$= -(P^2 - 2P + h)$$
$$= -(P-P_1)(P-P_2)\,.$$
$$P_1, P_2 = \frac{2 \pm \sqrt{4-4h}}{2}$$
$$= 1 \pm \sqrt{1-h}$$

Case 1: subcritical harvesting
$0 < h < 1$

Case 2. Critical harvesting

$h = 1$

$P'(t) = -(P-1)^2$
$= -(P^2 - 2P + 1)$

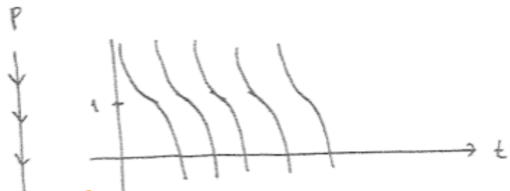
Case 3 Over harvesting

$h > 1$

complex roots.

$P'(t) = -(P^2 - 2P + h)$
$= -[(P-1)^2 + (h-1)]$
$< 0.$

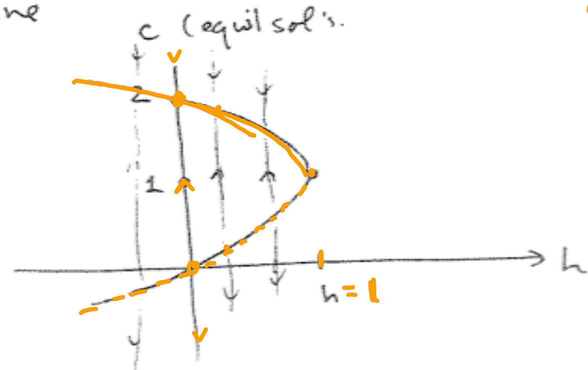
$P^2 - 2P + 1 + (h-1) = P^2 - 2P + h$ ✓



This model gives a plausible explanation for why many fisheries have "unexpectedly" collapsed in modern history. If $h < 1$ but near 1 and something perturbs the system a little bit (a bad winter, or a slight increase in fishing pressure), then the population and/or model could suddenly shift so that $P(t) \rightarrow 0$ very quickly.

Here's one picture that summarizes all the cases - you can think of it as collection of the phase diagrams for different fishing pressures h. The upper half of the parabola represents the stable equilibria, and the lower half represents the unstable equilibria. Diagrams like this are called "bifurcation diagrams". In the sketch below, the point on the h- axis should be labeled $h = 1$, not h. What's shown is the parabola of equilibrium solutions, $c = 1 \pm \sqrt{1 - h}$, i.e. $2c - c^2 - h = 0$, i.e. $h = c(2 - c)$.

"bifurcation diagram" of equilibrium solns in the h-c plane



graph of equil solns as a fcn of h.

roots $P = c$ of

$P^2 - 2P + h = 0$

$c = 1 \pm \sqrt{1-h}$.

$h = -P^2 + 2P$