Solve $\begin{cases} P'(t) = kP(M-P) \\ P(0) = P_0 \end{cases}$ $\frac{dP}{dt} = -kP(P-M)$

$$\int \frac{dP}{P(P-M)} = \int -k\,dt$$

$$\frac{1}{P(P-M)} = \frac{A}{P} + \frac{B}{P-M} = \frac{A(P-M)+BP}{P(P-M)} \qquad A = -\frac{1}{M},\ B = \frac{1}{M}$$

$$1 = A(P-M) + BP$$

@ $P=0$: $1 = -AM \Rightarrow A = -\frac{1}{M}$

@ $P=M$: $1 = BM \Rightarrow B = \frac{1}{M}$

$$\frac{1}{P(P-M)} = \frac{1}{M}\left(\frac{1}{P-M} - \frac{1}{P}\right)$$

$$M\cdot\cancel{\tfrac{1}{M}}\int\left(\frac{1}{P-M} - \frac{1}{P}\right)dP = M\int -k\,dt$$

$$\ln|P-M| - \ln|P| = -Mkt + C$$

$$\ln\left|\frac{P-M}{P}\right| = -Mkt + C_1$$

$$\left|\frac{P-M}{P}\right| = e^{-Mkt}e^{C_1}$$

$$\frac{P-M}{P} = Ce^{-Mkt} \leftarrow \text{find } C \text{ at } t=0: \quad \frac{P_0-M}{P_0} = C$$

mult by P: $P - M = PCe^{-Mkt}$

collect P's $P(t)\left[1 - Ce^{-Mkt}\right] = M$

$$P(t) = \frac{M}{1 - Ce^{-Mkt}}$$

$$P(t) = \frac{M}{1 - \frac{P_0 - M}{P_0}e^{-Mkt}} \cdot \frac{P_0}{P_0}$$

$$\boxed{P(t) = \frac{MP_0}{P_0 - (P_0 - M)e^{-Mkt}}}$$

$$= \frac{MP_0}{P_0 + (M - P_0)e^{-Mkt}}.$$

(If $P_0 < 0$, there's a vertical asymptote $t>0$ when $P_0 + (M-P_0)e^{-Mkt} = 0$)

$x'(t) = a(x-b)(x-c)$

$$\int \frac{dx}{(x-b)(x-c)} = \int a\,dt$$

$$\frac{1}{(x-b)(x-c)} = \frac{A}{x-b} + \frac{B}{x-c}$$

...

shortcut:

$$\frac{1}{(x-b)(x-c)} = \frac{1}{b-c}\left(\frac{1}{x-b} - \frac{1}{x-c}\right)$$

$$\frac{(\cancel{x}-c) - (\cancel{x}-b)}{(x-b)(x-c)}$$

$$\frac{1}{b-c}\,\frac{b-c}{(x-b)(x-c)} = \frac{1}{(x-b)(x-c)}$$

same

$$\boxed{\frac{x-b}{x-c} = \ldots}$$

compare to phase diagram & slope field predictions.

$P_0 > 0$: $\lim_{t\to\infty} P(t) = \frac{MP_0}{P_0} = \underline{\underline{M}}$

check denom is never 0 for $t \geq 0$

$0 < P_0 < M$ then each term in denom > 0 $(M - P_0 > 0)$. ✓

$P_0 > M$: $P_0 > |M - P_0| \Rightarrow$ denom > 0.

Exercise 2: Solve the logistic DE IVP

$$\frac{dP}{dt} = P' = k\,P(M-P)$$
$$P(0) = P_0$$

via separation of variables. Verify that the solution formula is consistent with the slope field and phase diagram discussion from exercise 1. Hint: You should find that

$$P(t) = \frac{MP_0}{(M-P_0)e^{-Mkt} + P_0}\ .$$

Solution (we will work this out step by step in class):

$$\frac{dP}{P(P-M)} = -k\,dt$$

By partial fractions,

$$\frac{1}{P(P-M)} = \frac{1}{M}\left(\frac{1}{P-M} - \frac{1}{P}\right).$$

Use this expansion and multiply both sides of the separated DE by M to obtain

$$\left(\frac{1}{P-M} - \frac{1}{P}\right)dP = -kM\,dt\ .$$

Integrate:

$$\ln|P-M| - \ln|P| = -Mkt + C_1$$
$$\ln\left|\frac{P-M}{P}\right| = -Mkt + C_1$$

exponentiate:

$$\left|\frac{P-M}{P}\right| = C_2\,e^{-Mkt}$$

Since the left-side is continuous

$$\frac{P-M}{P} = C\,e^{-Mkt} \quad (C = C_2 \text{ or } C = -C_2)$$

(At $t = 0$ we see that

$$\frac{P_0 - M}{P_0} = C\ .)$$

Now, solve for $P(t)$ by multiplying both sides of of the second to last equation by $P(t)$:

$$P - M = Ce^{-Mkt}P$$

Collect $P(t)$ terms on left, and add M to both sides:

$$P - Ce^{-Mkt}P = M$$
$$P\left(1 - Ce^{-Mkt}\right) = M$$
$$P = \frac{M}{1 - Ce^{-Mkt}}\ .$$

Plug in C and simplify:

$$P = \frac{M}{1 - \left(\frac{P_0 - M}{P_0}\right)e^{-Mkt}} = \frac{MP_0}{P_0 - (P_0 - M)e^{-Mkt}}$$

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-Mkt}}\ .$$

Finally, because $\lim_{t \to \infty} e^{-Mkt} = 0$, we see that

$$\lim_{t \to \infty} P(t) = \frac{MP_0}{P_0} = M \text{ as expected.}$$

Note: If $P_0 > 0$ the denominator stays positive for $t \geq 0$, so we know that the formula for $P(t)$ is a differentiable function for all $t > 0$. (If the denominator became zero, the function would blow up at the corresponding vertical asymptote.) To check that the denominator stays positive check that (i) if $P_0 < M$ then the denominator is a sum of two positive terms; if $P_0 = M$ the separation algorithm actually fails because you divided by 0 to get started but the formula actually recovers the constant equilibrium solution $P(t) \equiv M$; and if $P_0 > M$ then $|M - P_0| < P_0$ so the second term in the denominator can never be negative enough to cancel out the positive P_0 , for $t > 0$.)

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-Mkt}} \ .$$

Finally, because $\lim_{t \to \infty} e^{-Mkt} = 0$, we see that

$$\lim_{t \to \infty} P(t) = \frac{MP_0}{P_0} = M \text{ as expected.}$$

Note: If $P_0 > 0$ the denominator stays positive for $t \geq 0$, so we know that the formula for $P(t)$ is a differentiable function for all $t > 0$. (If the denominator became zero, the function would blow up at the corresponding vertical asymptote.) To check that the denominator stays positive check that (i) if $P_0 < M$ then the denominator is a sum of two positive terms; if $P_0 = M$ the separation algorithm actually fails because you divided by 0 to get started but the formula actually recovers the constant equilibrium solution $P(t) \equiv M$; and if $P_0 > M$ then $|M - P_0| < P_0$ so the second term in the denominator can never be negative enough to cancel out the positive P_0 , for $t > 0$.)

Question: You have a couple of homework problems where you are asked to find solutions $x(t)$ to differential equations of the form

$$x'(t) = a(x - b) \cdot (x - c).$$

How would you proceed?

Application

The Belgian demographer P.F. Verhulst introduced the logistic model around 1840, as a tool for studying human population growth. Our text demonstrates its superiority to the simple exponential growth model, and also illustrates why mathematical modelers must always exercise care, by comparing the two models to actual U.S. population data.

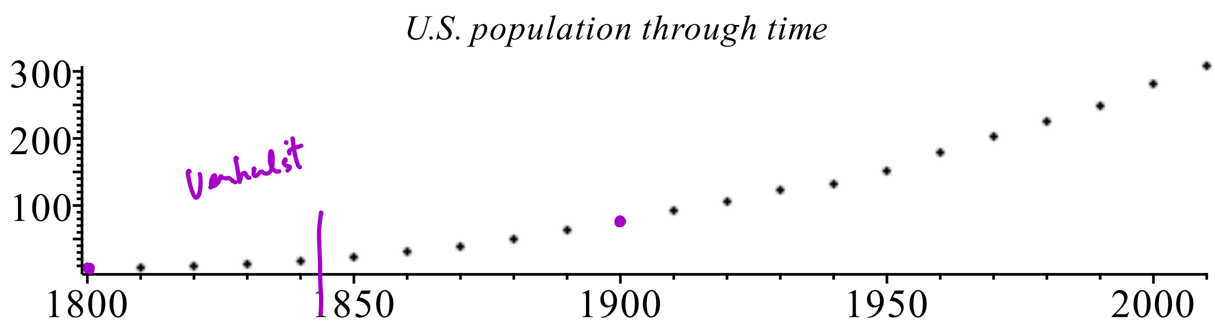
```
> restart : # clear memory
  Digits := 5 : #work with 5 significant digits
```

P_0

```
> pops := [[1800, 5.3], [1810, 7.2], [1820, 9.6], [1830, 12.9],
     [1840, 17.1], [1850, 23.2], [1860, 31.4], [1870, 38.6],
     [1880, 50.2], [1890, 63.0], [1900, 76.2], [1910, 92.2],
     [1920, 106.0], [1930, 123.2], [1940, 132.2], [1950, 151.3],
     [1960, 179.3], [1970, 203.3], [1980, 225.6], [1990, 248.7],
     [2000, 281.4], [2010, 308.]] : #I added 2010 - between 306-313
      # I used shift-enter to enter more than one line of information
      # before executing the command.
```

U.S. populations since 1800

```
> with(plots) : # plotting library of commands
  pointplot(pops, title = `U.S. population through time`);
```




```
>
```

Unlike Verhulst, the book uses data from 1800, 1850 and 1900 to get constants in our two models. We let t=0 correspond to 1800.

Exponential Model: For the exponential growth model $P(t) = P_0\, e^{r\,t}$ we use the 1800 and 1900 data to get values for P_0 and r :

need P_0 r

pop in 1900

```
> P0 := 5.308;
  solve(P0·exp(r·100) = 76.212, r);
```

$$P0 := 5.308$$

$$0.026643 \tag{1}$$

```
> P1 := t→5.308· exp(.02664· t);#exponential model -eqtn (9) page 83
```

$$P1 := t \rightarrow 5.308\, e^{0.02664\, t} \tag{2}$$

```
>
```

Logistic Model: We get P_0 from 1800, and use the 1850 and 1900 data to find k and M :

```
> P2 := t→M·P0/(P0 + (M−P0)·exp(−M·k·t)); # logistic solution we worked out
```

$$P2 := t \rightarrow \frac{M\,P0}{P0 + (M - P0)\,\mathrm{e}^{-M k t}} \tag{3}$$

$\rightarrow P_0,\ P(100)$
$P(50)$

```
> solve({P2(50) = 23.192, P2(100) = 76.212}, {M, k});
```

$$\{M = 188.12,\ k = 0.00016772\} \tag{4}$$

```
> M := 188.12;
  k := .16772e-3;
  P2(t); #should be our logistic model function,
     #equation (11) page 84.
```

$$M := 188.12$$

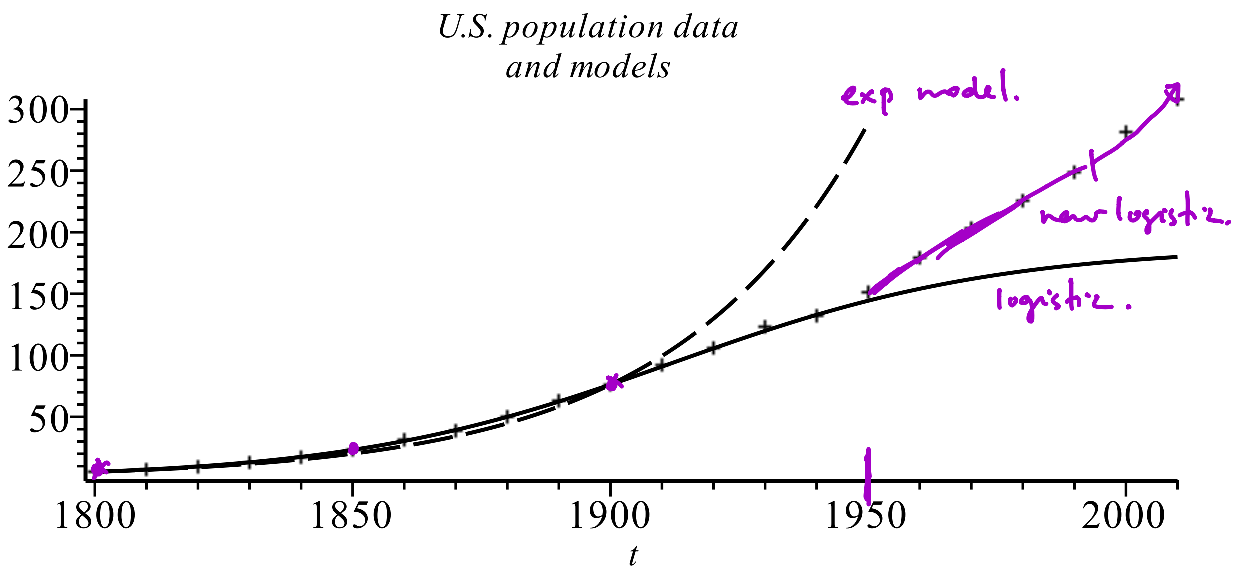
$$k := 0.00016772$$

$$\frac{998.54}{5.308 + 182.81\,\mathrm{e}^{-0.031551\,t}} \tag{5}$$

```
>
```

Now compare the two models with the real data, and discuss. The exponential model takes no account of the fact that the U.S. has only finite resources. Any ideas on why the logistic model begins to fail (with our parameters) around 1950?

```
> plot1 := plot(P1(t−1800), t = 1800..1950, color = black, linestyle = 3) :
   #this linestyle gives dashes for the exponential curve
  plot2 := plot(P2(t−1800), t = 1800..2010, color = black) :
  plot3 := pointplot(pops, symbol = cross) :
  display({plot1, plot2, plot3}, title = `U.S. population data
  and models`);
```




```
>
```

technology changed M.
life expectancies went up; δ went down
fertility rate went up.

```
>
```

Wed

- finish logistic DE discussion Monday notes
- then start today's notes §2.2

Math 2280-01
Wed Jan ~~28~~ 25
2.2: Autonomous Differential Equations.

Recall, that a general first order DE for $x = x(t)$ is written in standard form as

$$x' = f(t, x) \,,$$

which is shorthand for $x'(t) = f(t, x(t))$.
Definition: If the slope function f only depends on the value of $x(t)$, and not on t itself, then we call the first order differential equation *autonomous*:

$$x' = f(x) \,.$$

$P'(t) = kP(M-P)$

how fast x is changing only depends on value of x.

Example: The logistic DE, $P' = k\,P(M - P)$ is an autonomous differential equation for $P(t)$.

Definition: Constant solutions $x(t) \equiv c$ to autonomous differential equations $x' = f(x)$ are called *equilibrium solutions*. Since the derivative of a constant function $x(t) \equiv c$ is zero, the values c of equilibrium solutions are exactly the roots c to $f(c) = 0$.

const soltn to $x'(t) = f(x)$
$x(t) \equiv c$: LHS $=0 \Rightarrow$ RHS $=0$
$f(c) = 0$

Example: The functions $P(t) \equiv 0$ and $P(t) \equiv M$ are the equilibrium solutions for the logistic DE.

Exercise 1: Find the equilibrium solutions of

1a) $x'(t) = 3\,x - x^2$ $= x(3-x)$

equil soltns $x(t) \equiv 0$, $x(t) \equiv 3$.
equil pts $x = 0, 3$

1b) $x'(t) = x^3 + 2\,x^2 + x$ $= x(x^2+2x+1)$
$= x(x+1)^2$

equil sol's $x \equiv 0, -1$

1c) $x'(t) = \sin(x)$.

$x = 0, \pm\pi, \pm 2\pi, \ldots$
$x = k\pi \quad k \in \mathbb{Z}$

start close / stay close

Def: Let $x(t) \equiv c$ be an equilibrium solution for an autonomous DE. Then

· c is a *stable* equilibrium solution if solutions with initial values close enough to c stay close to c.

There is a precise way to say this, but it requires quantifiers: For every $\varepsilon > 0$ there exists a $\delta > 0$ so that for solutions with $|x(0) - c| < \delta$, we have $|x(t) - c| < \varepsilon$ for all $t > 0$.

· c is an *unstable* equilibrium if it is not stable.

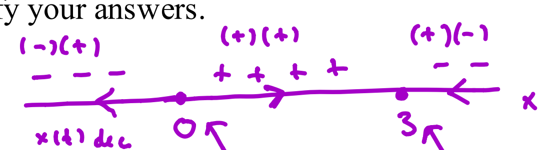
· c is an *asymptotically stable* equilibrium solution if it's stable and in addition, if $x(0)$ is close enough to c, then $\lim_{t \to \infty} x(t) = c$, i.e. there exists a $\delta > 0$ so that if $|x(0) - c| < \delta$ then $\lim_{t \to \infty} x(t) = c$. (Notice that this means the horizontal line $x = c$ will be an *asymptote* to the solution graphs $x = x(t)$ in these cases.)

note if slope fcn $f(x)$ is cont, & $\frac{\partial f}{\partial x}$ is cont. then by uniqueness theorem no two solution graphs can even touch. $x'(t) = f(x)$

Exercise 2: Use phase diagram analysis to guess the stability of the equilibrium solutions in Exercise 1. For (a) you've worked out a solution formula already, so you'll know you're right. For (b), (c), use the Theorem on the next page to justify your answers.

2a) $x'(t) = 3x - x^2$

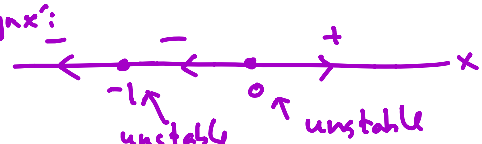
$= x(3-x)$



phase diagram

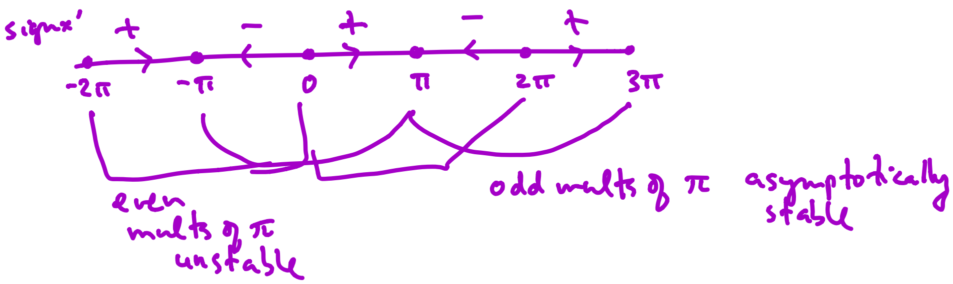
2b) $x'(t) = x^3 + 2x^2 + x$

$= x(x+1)^2$



2c) $x'(t) = \sin(x)$.

$= 0$ @ $x = k\pi$



Theorem: Consider the autonomous differential equation

$$x'(t) = f(x).$$

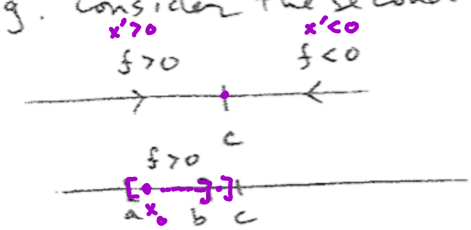
with $f(x)$ and $\frac{\partial}{\partial x} f(x)$ continuous (so local existence and uniqueness theorems hold). Let $f(c) = 0$, i.e. $x(t) \equiv c$ is an equilibrium solution. Suppose c is an *isolated zero* of f, i.e. there is an open interval containing c so that c is the only zero of f in that interval. The the stability of the equilibrium solution c can is completely determined by the local phase diagrams:

$$
\begin{array}{rlcccl}
sign(f): & \underset{c}{- - - - 0 + + +} & \Rightarrow & \leftarrow\leftarrow\leftarrow c \rightarrow\rightarrow\rightarrow & \Rightarrow & c \text{ is unstable} \\
sign(f): & + + + 0 - - - - & \Rightarrow & \rightarrow\rightarrow\rightarrow c \leftarrow\leftarrow\leftarrow & \Rightarrow & c \text{ is asymptotically stable} \\
sign(f): & + + + 0 + + + & \Rightarrow & \rightarrow\rightarrow\rightarrow c \rightarrow\rightarrow\rightarrow & \Rightarrow & c \text{ is unstable (half stable)} \\
sign(f): & - - - - 0 - - - - & \Rightarrow & \leftarrow [\leftarrow \bullet \leftarrow] c \leftarrow\leftarrow\leftarrow & \Rightarrow & c \text{ is unstable (half stable)}
\end{array}
$$

You can actually prove this Theorem with calculus!! (want to try?)

Here's why!

e.g. consider the second case



f cont; $f > 0$ on subinterval $[a,b]$

$\Rightarrow f \geq \delta > 0$ on $[a,b]$

(extreme value thm from calculus, f attains its minimum)

$x'(t) = f(x) \geq \delta$ as long as $x \in [a,b]$

$\Rightarrow x'(t) \geq \delta$ as long as $x(t) \in [a,b]$

$\Rightarrow x(t)$ stays in this interval for time interval at most $\underbrace{\frac{b-a}{\delta}}$

speed · time = dist

time = $\frac{\text{dist}}{\text{speed}}$.

$\Rightarrow \lim_{t \to \infty} x(t) = c$

(because we can pick the right endpoint b as close to c ($b<c$) as we want, to ensure $x(t)$ eventually gets as close as we want to the value c)

Exercise 3) Use the chain rule to check that if $x(t)$ solves the autonomous DE

$$x'(t) = f(x)$$

Then $X(t) := x(t - c)$ solves the same DE. What does this say about the geometry of representative solution graphs to autonomous DEs? Have we already noticed this?

Further application: Doomsday-extinction. With different hypotheses about fertility and mortality rates, one can arrive at a population model which looks like logistic, except the right hand side is the opposite of what it was in that case:

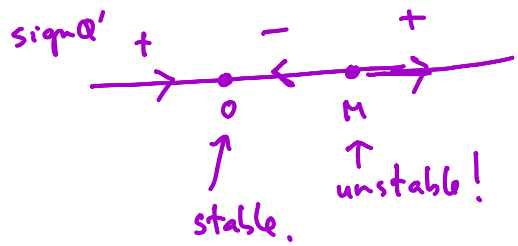
Logistic: $P'(t) = -a\,P^2 + b\,P$ $\quad kP(M-P)$

Doomsday-extinction: $Q'(t) = a\,Q^2 - b\,Q$

For example, suppose that the chances of procreation are proportional to population density (think alligators or crickets), i.e. the fertility rate $\beta = a\,Q(t)$, where $Q(t)$ is the population at time t. Suppose the morbidity rate is constant, $\delta = b$. With these assumptions the birth and death rates are $a\,Q^2$ and $-b\,Q$ .... which yields the DE above. In this case factor the right side:

$$Q'(t) = a\,Q\left(Q - \frac{b}{a}\right) = k\,Q(Q - M). \qquad kQ(Q-M)$$

Exercise 4a) Construct the phase diagram for the general doomsday-extinction model and discuss the stability of the equilbrium solutions.



$Q_0 > M$ "doomsday" in finite time

$Q_0 < M$ $Q(t) \to 0$ as $t \to \infty$

"extinction"