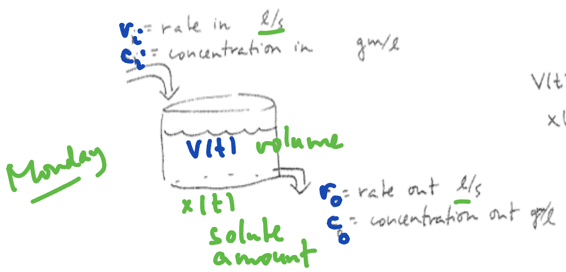
An extremely important class of modeling problems that lead to linear DE's involve input-output models. These have diverse applications ranging from bioengineering to environmental science. For example, The "tank" below could actually be a human body, a lake, or a pollution basin, in different applications.

For the present considerations, consider a tank holding liquid, with volume $V(t)$ (e.g units l). Liquid flows in at a rate r_i (e.g. units $\frac{l}{s}$ — volume/time), and with solute concentration c_i (e.g. units $\frac{gm}{l}$ — mass/volume). Liquid flows out at a rate r_o, and with concentration c_0. We are attempting to model the volume $V(t)$ of liquid and the amount of solute $x(t)$ (e.g. units gm — mass) in the tank at time t, given $V(0) = V_0,\ x(0) = x_0$. We assume the solution in the tank is well-mixed, so that we can treat the concentration as uniform throughout the tank, i.e.

$$c_o = \frac{x(t)}{V(t)} \quad \frac{gm}{l}.$$

concentration everywhere in tank is the same i.e. average conc. $= \frac{x(t)}{V(t)}$ $\frac{\text{mass}}{\text{volume}}$.

See the diagram below.



$V(t)$ = volume in tank at time t (l)

$x(t)$ = amount of solute in tank (gm)

$c(t) = \frac{x(t)}{V(t)}$ $\frac{g}{l}$ (average) concentration in tank

Exercise 4: Under these assumptions use your modeling ability and Calculus to derive the following differential equations for $V(t)$ and $x(t)$:

a) The DE for $V(t)$, which we can just integrate:

$$\boxed{V'(t) = r_i - r_0}.$$

$\uparrow \frac{\text{vol}}{\text{time}}$ $\uparrow \frac{\text{vol}}{\text{time}}$

$\int_0^t V'(\tau)d\tau = \int_0^t r_i(\tau) - r_o(\tau)\, d\tau$ §1.2

so $V(t) = V_0 + \int_0^t r_i(\tau) - r_0(\tau)\, d\tau$

in time Δt

$\Delta V \approx \underbrace{r_i}_{\text{vol/time}} \underbrace{\Delta t}_{\text{time}} - r_o \Delta t$ vol.

$\frac{\Delta V}{\Delta t} \approx r_i - r_o$

$V'(t) = r_i - r_o$ $\frac{\text{vol}}{\text{time}}$

b) The linear DE for $x(t)$.

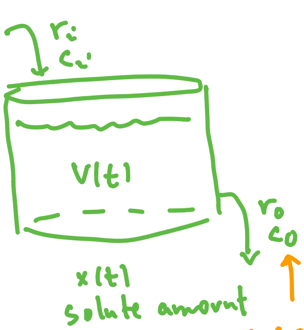
$$\boxed{x'(t) = r_i c_i - r_o c_o} = r_i c_i - r_o \frac{x}{V}.$$

$$\boxed{x'(t) + \frac{r_o}{V} x(t) = r_i c_i}$$

e.g. $r_i = 10 \frac{l}{\text{min}}$, $c_i = \frac{2g}{l}$ → solute coming into the tank $20 \frac{g}{\text{min}}$

$r_o = 5 \frac{l}{\text{min}}$, $c_o = \frac{x(t)}{V(t)}$ → solute leaving $30 \frac{g}{\text{min}}$

$= 6 \frac{g}{l}$



we assume solute is uniformly mixed so c_o = average conc. $= \frac{x(t)}{V(t)}$

in time increment Δt estimate Δx

$\Delta x \cong$ how much comes in $-$ how much goes out

$=$ (vol. comes in) $\cdot$ amt/vol. $-$ (vol. goes out) $\cdot \frac{\text{amt}}{\text{vol.}}$

$\Delta x \cong (r_i \cdot \Delta t)\, c_i - (r_o \cdot \Delta t) \frac{x(t)}{V(t)}$

$\uparrow \frac{\text{vol}}{\text{time}}$ $\uparrow$ time

Linear: $x'(t) + \frac{r_o}{V(t)} x(t) = r_i c_i$ (const, const, const)

$\div \Delta t$: $\frac{\Delta x}{\Delta t} \approx r_i c_i - r_o \frac{x}{V}$

let $\Delta t \to 0$ $\boxed{x'(t) = r_i c_i - r_o \frac{x}{V}}$

Often (but not always) the tank volume remains constant, i.e. $r_i = r_o$ = const. If the incoming concentration c_i is also constant, then the IVP for solute amount is

$$x' + a\,x = b$$
$$x(0) = x_0$$

where a, b are constants.

Exercise 5: The constant coefficient initial value problem above will recur throughout the course in various contexts, so let's solve it now. Hint: We will check our answer with Maple first, and see that the solution is

$$x(t) = \frac{b}{a} + \left(x_o - \frac{b}{a}\right)e^{-a\,t}.$$

use linear algorithm

$$x'(t) + \underbrace{a}_{P(t)}\, x(t) = \underbrace{b}_{Q(t)}$$

① $\int P(t)\,dt = \int a\,dt = at$

I.F. $e^{\int P(t)\,dt} = e^{at}$

② $e^{at}(x' + ax) = e^{at}\, b$

$\frac{d}{dt}(e^{at}x) = e^{at}\, b$

$\frac{d}{dt} e^{at} x = a e^{at} x + e^{at} x'$

LHS of DE: $e^{at}x' + e^{at}ax$

③ integrate: $e^{at}x = \int b e^{at}\,dt$

$e^{at}x = \frac{b}{a}e^{at} + C$

④ $\div e^{at}$ $x = \frac{b}{a} + Ce^{-at}$

IVP $x(0) = x_0 = \frac{b}{a} + C \Rightarrow C = x_0 - \frac{b}{a}$

$$\boxed{x(t) = \frac{b}{a} + \left(x_0 - \frac{b}{a}\right)e^{-at}}$$

Exercise 6 (taken from section 1.5 of text) Solve the following pollution problem IVP, to answer the follow-up question: Lake Huron has a relatively constant concentration for a certain pollutant. Since Lake Huron is the primary water source for Lake Erie, this is also the usual pollutant concentration in Lake Erie Due to an industrial accident, however, Lake Erie has suddenly obtained a concentration five times as large. Lake Erie has a volume of 480 km^3, and water flows into and out of Lake Erie at a rate of 350 km^3 per year. Essentially all of the in-flow is from Lake Huron (see below). We expect that as time goes by, the water from Lake Huron will flush out Lake Erie. Assuming that the pollutant concentration is roughly the same everywhere in Lake Erie, about how long will it be until this concentration is only twice the original background concentration from Lake Huron?

"c" $5c$ $2c$

http://www.enchantedlearning.com/usa/statesbw/greatlakesbw.GIF

$$\begin{cases} x' + ax = b \\ x(0) = x_0 \end{cases}$$

soln $x(t) = \frac{b}{a} + Ce^{-at}$, $C = (x_0 - \frac{b}{a})$

a) Set up the initial value problem. Maybe use symbols c for the background concentration (in Huron),
$V = 480\ km^3$
$r = 350\ \frac{km^3}{y}$

$r_i = r_o = r = 350\ km^3/\text{year}$
$\Rightarrow V = \text{const}$

b) Solve the IVP, and then answer the question.

Let $x(t)$ = pollutant amt in Lake Erie.

$x'(t) = r_i c_i - r_o c_o$ } well-mixed hypothesis

$$\begin{cases} x'(t) = rc - r\,\frac{x}{V} \\ x(0) = 5c\cdot V \end{cases}$$

initial concent.

$$\begin{cases} x' + \frac{r}{V}x = rc \\ x(0) = 5cV \end{cases}$$

transcribe soln $\Longrightarrow$

$a = \frac{r}{V}$
$b = rc$

$$x(t) = \frac{b}{a} + Ce^{-at} = \frac{rc}{r/V} + (5cV - cV)e^{-\frac{r}{V}t}$$

$$\boxed{x(t) = cV + 4cVe^{-\frac{r}{V}t}}$$

answer Q:
when concent $= 2c$
$$x(t) = 2\cancel{cV} = \cancel{cV} + 4\cancel{cV}e^{-\frac{r}{V}t}$$
$$2 = 1 + 4e^{-r/V\,t}$$

$$.25 = e^{-\frac{r}{V}t}$$

$$\ln(.25) = -\frac{r}{V}t$$

$$t = \frac{-V}{r}\ln(.25) = -\frac{480}{350}\ln(.25)$$

$$\approx 1.9 \text{ years}$$

Math 2280-001
Week 3: Jan 23-27, sections 2.1-2.3

Office hours (for 2250, 2280) T, W 4:30-6:00 LCB 218 conference room
expect more 2250 students on T
also after class LCB 204 MWF

Mon Jan 23
2.1 Improved population models.

- Finish any remaining material from Friday.

- Let $P(t)$ be a population at time t. Let's call them "people", although they could be other biological organisms, decaying radioactive elements, accumulating dollars, or even molecules of solute dissolved in a liquid at time t (2.1.23). Consider:

$B(t)$, birth rate (e.g. $\frac{people}{year}$);

$\beta(t) := \frac{B(t)}{P(t)}$, fertility rate ($\frac{people}{year}$ per *person*) $B = \beta(t)P$

$\mathrm{D}(t)$, death rate (e.g. $\frac{people}{year}$);

$\delta(t) := \frac{\mathrm{D}(t)}{P(t)}$, mortality rate ($\frac{people}{year}$ per *person*) $D(t) = \delta(t)P$

Then in a closed system (i.e. no migration in or out) we can write the governing DE two equivalent ways:

$$P'(t) = B(t) - \mathrm{D}(t) \quad + r_i - r_o$$
$$P'(t) = (\beta(t) - \delta(t))P(t).$$

Model 1: constant fertility and mortality rates, $\beta(t) \equiv \beta_0 \geq 0,\ \delta(t) \equiv \delta_0 \geq 0$, constants.

$$\Rightarrow P' = (\beta_0 - \delta_0)P = kP.$$

This is our familiar exponential growth/decay model, depending on whether $k > 0$ or $k < 0$.

Model 2: population fertility and mortality rates only depend on population P, but they are not constant:

$$\beta = \beta_0 + \beta_1 P$$
$$\delta = \delta_0 + \delta_1 P$$

$\beta_0 - \delta_0$ dominates $(\beta_1 - \delta_1)P$ inside the parentheses

with $\beta_0, \beta_1, \delta_0, \delta_1$ constants. This implies

$$P' = (\beta - \delta)P = ((\beta_0 + \beta_1 P) - (\delta_0 + \delta_1 P))P$$
$$= ((\beta_0 - \delta_0) + (\beta_1 - \delta_1)P)P.$$

when P is small, the

For viable populations, $\beta_0 > \delta_0$. For a sophisticated (e.g. human) population we might also expect $\beta_1 < 0$, and resource limitations might imply $\delta_1 > 0$. With these assumptions, and writing $\beta_1 - \delta_1 = -a < 0$, $\beta_0 - \delta_0 = b > 0$ one obtains the logistic differential equation:

$\hat{0}\ -(+) =$

$$P' = (b - aP)P$$
$$P' = -aP^2 + bP, \text{ or equivalently}$$
$$P' = aP\left(\frac{b}{a} - P\right) = kP(M - P).$$

$k = a > 0,\ M = \frac{b}{a} > 0$. (One can consider other cases as well.)

Exercise 1: Discuss qualitative features of the slope field for the logistic differential equation for $P = P(t)$:

$$P' = k\,P(M - P)$$

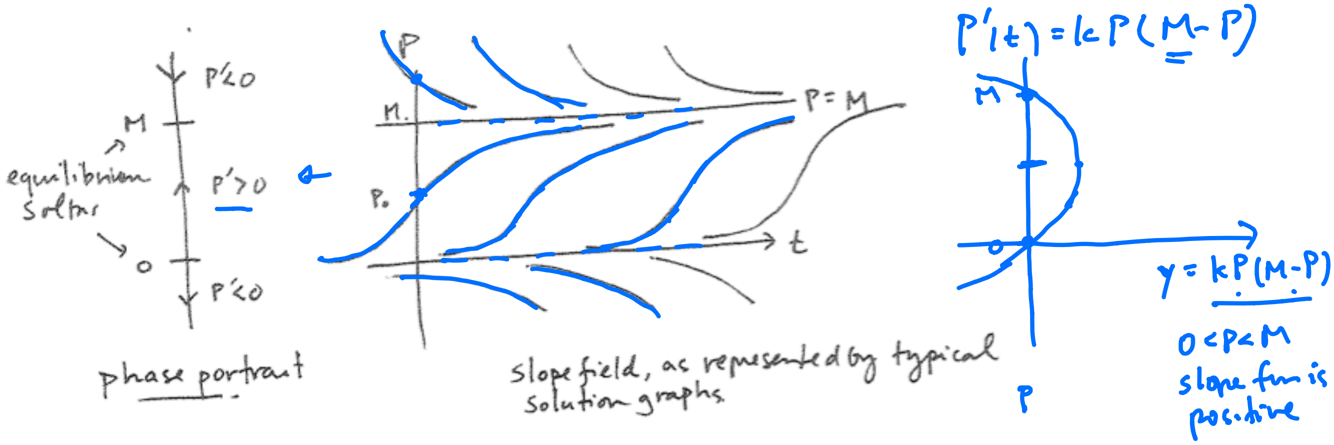
$$\frac{dP}{(P)(P-M)} = k\,dt$$

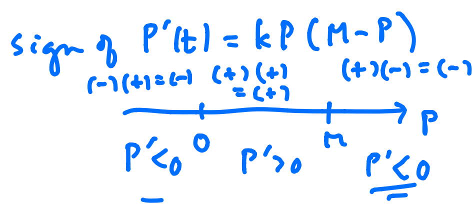
a) There are two constant ("equilibrium") solutions. What are they?

$P(t) \equiv M$
$P(t) \equiv 0$

slope fn, only depends on P, not on t

b) Evaluate the sign and magnitude of the slope function $f(P, t) = k\,P(M - P)$, in order to understand and be able to recreate the two diagrams below. One is a qualititative picture of the slope field, in the $t - P$ plane. The diagram to the left of it, called the phase diagram, is just a P number line with arrows indicating whether $P(t)$ is increasing or decreasing on the intervals between the constant solutions.





c) When discussing the logistic equation, the value M is called the "carrying capacity" of the (ecological or other) system. Discuss why this is a good way to describe M. Hint: if $P(0) = P_0 > 0$, and $P(t)$ solves the logistic equation, what is the apparent value of $\lim_{t \to \infty} P(t)$? $= M$

Math 2280-001 Quiz 2 January 20, 2017
SOLUTIONS

1) Consider the initial value problem

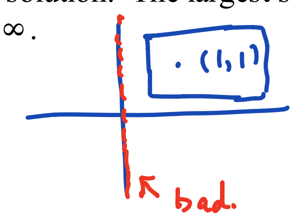
$$y'(x) = \frac{y^2}{x}$$
$$y(1) = 1.$$

initial point $(x_0, y_0) = (1,1)$

a) *Use the existence-uniqueness theorem to show that there some open interval containing $x_0 = 1$ on which this initial value problem has a unique solution. (Hint: Recall, if the slope function $f(x, y)$ is continuous in a coordinate rectangle R having the initial point in its interior, then there exists at least one solution. If the partial derivative $\frac{\partial}{\partial y} f(x, y)$ is also continuous, then the solution is unique as long as its graph remains inside R.)*

(2 points)

solution: The slope function $\boxed{f(x, y) = \frac{y^2}{x}}$ is continuous except along the $y - axis$, i.e. $x = 0$. The same holds true for $\frac{\partial}{\partial y}\left(\frac{y^2}{x}\right) = \boxed{\frac{2y}{x}}$. So any coordinate rectangle that contains the initial point $(1, 1)$ and avoids the $x - axis$ suffices to show that the IVP has a unique solution. The largest such coordinate rectangle is the "right half plane", given by $x > 0$, $-\infty < y < \infty$.



b) *The differential equation in this problem is separable, so you can actually find a solution to the initial value problem above. Do so.*

(6 points)

$$\frac{dy}{dx} = \frac{y^2}{x} \quad (x \neq 0)$$
$$\frac{dy}{y^2} = \frac{1}{x}\,dx \quad (y \neq 0)$$
$$\int y^{-2}\,dy = \int \frac{1}{x}\,dx$$
$$-y^{-1} = \ln|x| + C$$
$$y(1) = 1 \Rightarrow -1 = 0 + C \Rightarrow \underline{C = -1}$$
$$-\frac{1}{y} = \ln|x| - 1 = \ln(x) - 1 \ \text{ since } x > 0$$
$$y = -\frac{1}{\ln(x) - 1} = \frac{1}{1 - \ln x}$$

y undefined @ $x = e$
because $1 - \ln x = 0$ there.
graph has vert. asymp. @ $x = e$.
graph vertical @ $x = 0$
(DE doesn't make sense there)

c) *What is the largest x - interval on which the solution to* ***b*** *is defined as a differentiable function? Explain.*

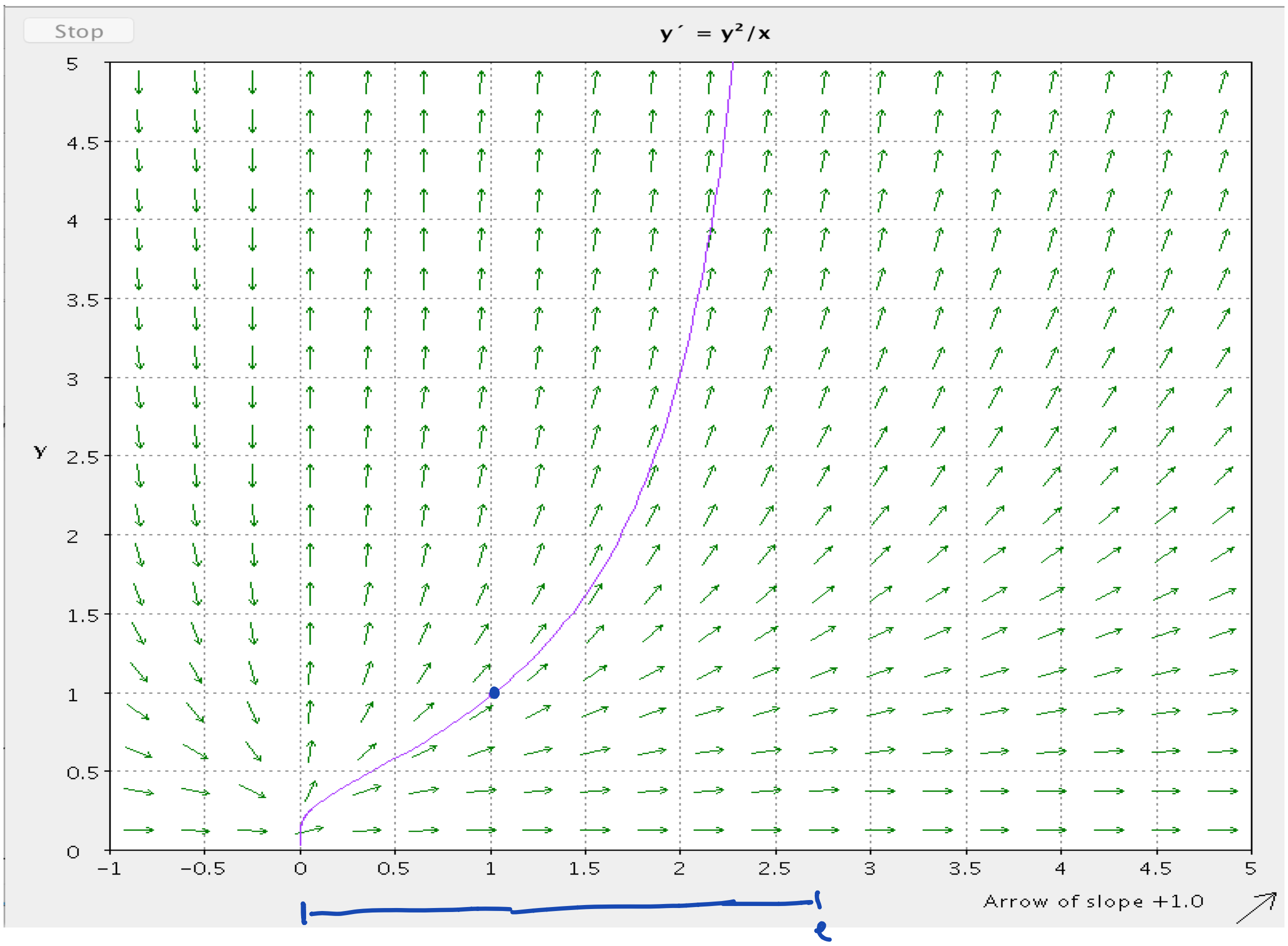
to IVP.

interval must contain $x_0 = 1$

(2 points)

solution: The largest interval containing $x_0 = 1$ on which $y(x)$ is not differentiable at $x = 0$, where y' blows up. ($\lim_{x \to 0+} y'(x) = \lim_{x \to 0+} \frac{y(x)^2}{x} = +\infty$). And, the graph has a vertical asymptote at $x = e$, where the denominator of $y(x)$ is zero. So the largest interval containing the initial point $x = 1$ on which the IVP solution exists is $0 < x < e$.

$(0, e)$


Stop
y´ = y²/x
y
x
Arrow of slope +1.0
e