Math 2280-001
Week 3: Jan 23-27, sections 2.1-2.3

Office hours (for 2250, 2280) T, W 4:30-6:00 LCB 218 conference room
expect more 2250 students on T
also after class LCB 204 MWF

Mon Jan 23
2.1 Improved population models.

- Finish any remaining material from Friday.

- Let $P(t)$ be a population at time t. Let's call them "people", although they could be other biological organisms, decaying radioactive elements, accumulating dollars, or even molecules of solute dissolved in a liquid at time t (2.1.23). Consider:

$B(t)$, birth rate (e.g. $\frac{people}{year}$);

$\beta(t) := \frac{B(t)}{P(t)}$, fertility rate ($\frac{people}{year}$ per *person*) $\qquad B = \beta(t)P$

$\mathrm{D}(t)$, death rate (e.g. $\frac{people}{year}$);

$\delta(t) := \frac{\mathrm{D}(t)}{P(t)}$, mortality rate ($\frac{people}{year}$ per *person*) $\qquad D(t) = \delta(t)P$

Then in a closed system (i.e. no migration in or out) we can write the governing DE two equivalent ways:

$$P'(t) = B(t) - \mathrm{D}(t) \quad + r_i - r_o$$
$$P'(t) = (\beta(t) - \delta(t))P(t) .$$

Model 1: constant fertility and mortality rates, $\beta(t) \equiv \beta_0 \geq 0$, $\delta(t) \equiv \delta_0 \geq 0$, constants.

$$\Rightarrow P' = (\beta_0 - \delta_0)P = kP .$$

This is our familiar exponential growth/decay model, depending on whether $k > 0$ or $k < 0$.

Model 2: population fertility and mortality rates only depend on population P, but they are not constant:

$$\beta = \beta_0 + \beta_1 P$$
$$\delta = \delta_0 + \delta_1 P$$

with $\beta_0, \beta_1, \delta_0, \delta_1$ constants. This implies

$$P' = (\beta - \delta)P = ((\beta_0 + \beta_1 P) - (\delta_0 + \delta_1 P))P$$
$$= ((\beta_0 - \delta_0) + (\beta_1 - \delta_1)P)P .$$

when P is small, the $\beta_0 - \delta_0$ dominates $(\beta_1 - \delta_1)P$ inside the parentheses.

For viable populations, $\beta_0 > \delta_0$. For a sophisticated (e.g. human) population we might also expect $\beta_1 < 0$, and resource limitations might imply $\delta_1 > 0$. With these assumptions, and writing $\beta_1 - \delta_1 = -a < 0$, $\beta_0 - \delta_0 = b > 0$ one obtains the logistic differential equation:

$\beta_1 - \delta_1$: $\hat{0} - (+) = -$

$$P' = (b - aP)P$$
$$P' = -aP^2 + bP \text{, or equivalently}$$
$$P' = aP\left(\frac{b}{a} - P\right) = kP(M - P).$$

$k = a > 0$, $M = \frac{b}{a} > 0$. (One can consider other cases as well.)

Exercise 1: Discuss qualitative features of the slope field for the logistic differential equation for $P = P(t)$:

$$P' = k\,P(M - P)\ .$$

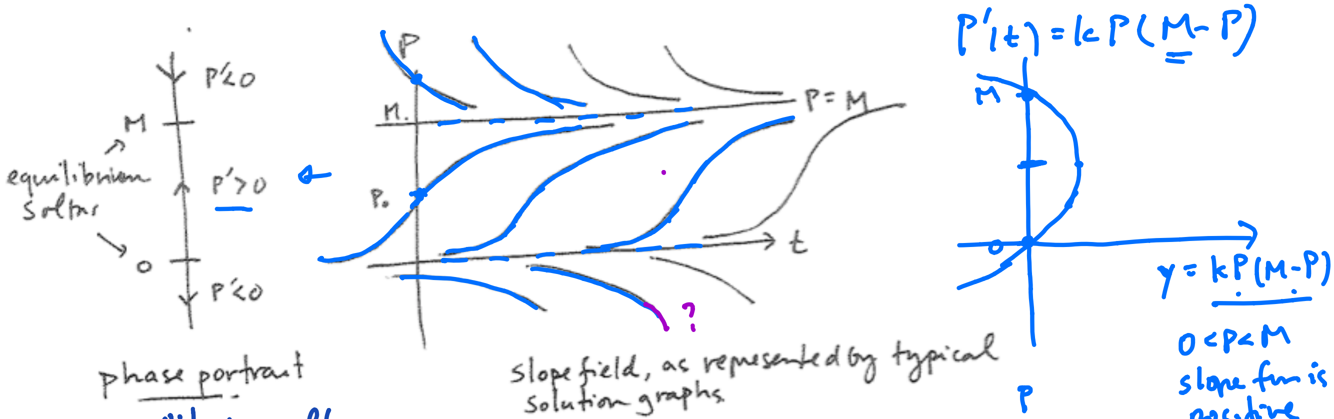
$$\frac{dP}{(P)(P-M)} = k\,dt$$

a) There are two constant ("equilibrium") solutions. What are they?

$P(t) \equiv M$
$P(t) \equiv 0$

slope fn, only depends on P, not on t

b) Evaluate the sign and magnitude of the slope function $f(P, t) = k\,P(M - P)$, in order to understand and be able to recreate the two diagrams below. One is a qualititative picture of the slope field, in the $t - P$ plane. The diagram to the left of it, called the phase diagram , is just a P number line with arrows indicating whether $P(t)$ is increasing or decreasing on the intervals between the constant solutions.



• equilibrium soltns
• arrows indicate whether P(t) is increasing or decreasing in between equil. pts.

sign of $P'(t) = kP(M-P)$

$(-)(+) = (-)$ $\quad (+)(+) = (+)$ $\quad (+)(-) = (-)$

$P'<0$ $\quad 0 \quad$ $P'>0$ $\quad M \quad$ $P'<0$ $\quad P$

c) When discussing the logistic equation, the value M is called the "carrying capacity" of the (ecological or other) system. Discuss why this is a good way to describe M . Hint: if $P(0) = P_0 > 0$, and $P(t)$ solves the logistic equation, what is the apparent value of $\lim_{t \to \infty} P(t)$? $= M$

Solve $\begin{cases} P'(t) = kP(M-P) \\ P(0) = P_0 \end{cases}$ $\quad \dfrac{dP}{dt} = -kP(P-M)$

$$\int \frac{dP}{P(P-M)} = \int -k\,dt$$

$$\frac{1}{P(P-M)} = \frac{A}{P} + \frac{B}{P-M} = \frac{A(P-M)+BP}{P(P-M)}$$

$A = -\frac{1}{M}$, $B = \frac{1}{M}$

$$1 = A(P-M) + BP$$

@ $P=0$: $1 = -AM \Rightarrow A = -\frac{1}{M}$

@ $P=M$: $1 = BM \Rightarrow B = \frac{1}{M}$

$$\frac{1}{P(P-M)} = \frac{1}{M}\left(\frac{1}{P-M} - \frac{1}{P}\right)$$

$$\cancel{M}\cdot\cancel{\frac{1}{M}}\int\left(\frac{1}{P-M} - \frac{1}{P}\right)dP = M\int -k\,dt$$

$$\ln|P-M| - \ln|P| = -Mkt + C$$

$$\ln\left|\frac{P-M}{P}\right| = -Mkt + C_1$$

e $\qquad e$

$$\left|\frac{P-M}{P}\right| = e^{-Mkt}e^{C_1}$$

$$\frac{P-M}{P} = Ce^{-Mkt} \leftarrow \text{find } C \text{ at } t=0: \quad \frac{P_0-M}{P_0} = C$$

mult by P: $P - M = PCe^{-Mkt}$

collect P's $\quad P(t)\left[1 - Ce^{-Mkt}\right] = M$

$$P(t) = \frac{M}{1 - Ce^{-Mkt}}$$

$$P(t) = \frac{M}{1 - \frac{P_0-M}{P_0}e^{-Mkt}} \cdot \frac{P_0}{P_0}$$

$$\boxed{P(t) = \frac{MP_0}{P_0 - (P_0-M)e^{-Mkt}}}$$

$$= \frac{MP_0}{P_0 + (M-P_0)e^{-Mkt}}.$$

(If $P_0 < 0$, there's a vertical asymptote $t > 0$ when $P_0 + (M-P_0)e^{-Mkt} = 0$)

$$x'(t) = a(x-b)(x-c)$$

$$\int \frac{dx}{(x-b)(x-c)} = \int a\,dt$$

$$\frac{1}{(x-b)(x-c)} = \frac{A}{x-b} + \frac{B}{x-c}$$

...

shortcut:

$$\frac{1}{(x-b)(x-c)} = \frac{1}{b-c}\left(\underbrace{\frac{1}{x-b} - \frac{1}{x-c}}_{\frac{(\cancel{x}-c)-(\cancel{x}-b)}{(x-b)(x-c)}}\right)$$

$$\frac{1}{b-c}\,\frac{b-c}{(x-b)(x-c)} = \frac{1}{(x-b)(x-c)}$$

same

$$\boxed{\frac{x-b}{x-c} = \ldots}$$

compare to phase diagram & slope field predictions.

$P_0 > 0$: $\lim_{t\to\infty} P(t) = \frac{MP_0}{P_0} = \underline{\underline{M}}$

check denom is never 0 for $t \geq 0$

$0 < P_0 < M$ then each term in denom > 0 $(M - P_0 > 0)$. ✓

$P_0 > M$: $P_0 > |M - P_0| \Rightarrow$ denom > 0.

Exercise 2: Solve the logistic DE IVP

$$\frac{dP}{dt} = P' = k\,P(M - P)$$
$$P(0) = P_0$$

via separation of variables. Verify that the solution formula is consistent with the slope field and phase diagram discussion from exercise 1. Hint: You should find that

$$P(t) = \frac{MP_0}{\left(M - P_0\right)e^{-Mkt} + P_0} \ .$$

Solution (we will work this out step by step in class):

$$\frac{dP}{P(P - M)} = -k\,dt$$

By partial fractions,

$$\frac{1}{P(P - M)} = \frac{1}{M}\left(\frac{1}{P - M} - \frac{1}{P}\right).$$

Use this expansion and multiply both sides of the separated DE by M to obtain

$$\left(\frac{1}{P - M} - \frac{1}{P}\right)dP = -kM\,dt \ .$$

Integrate:

$$\ln|P - M| - \ln|P| = -Mkt + C_1$$
$$\ln\left|\frac{P - M}{P}\right| = -Mkt + C_1$$

exponentiate:

$$\left|\frac{P - M}{P}\right| = C_2\,e^{-Mkt}$$

Since the left-side is continuous

$$\frac{P - M}{P} = C\,e^{-Mkt} \quad (C = C_2 \text{ or } C = -C_2)$$

(At $t = 0$ we see that

$$\frac{P_0 - M}{P_0} = C\ .)$$

Now, solve for $P(t)$ by multiplying both sides of of the second to last equation by $P(t)$:

$$P - M = Ce^{-Mkt}P$$

Collect $P(t)$ terms on left, and add M to both sides:

$$P - Ce^{-Mkt}P = M$$
$$P\left(1 - Ce^{-Mkt}\right) = M$$
$$P = \frac{M}{1 - Ce^{-Mkt}} \ .$$

Plug in C and simplify:

$$P = \frac{M}{1 - \left(\dfrac{P_0 - M}{P_0}\right)e^{-Mkt}} = \frac{MP_0}{P_0 - \left(P_0 - M\right)e^{-Mkt}}$$

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-Mkt}} .$$

Finally, because $\lim_{t \to \infty} e^{-Mkt} = 0$, we see that

$$\lim_{t \to \infty} P(t) = \frac{MP_0}{P_0} = M \text{ as expected.}$$

<u>Note:</u> If $P_0 > 0$ the denominator stays positive for $t \geq 0$, so we know that the formula for $P(t)$ is a differentiable function for all $t > 0$. (If the denominator became zero, the function would blow up at the corresponding vertical asymptote.) To check that the denominator stays positive check that (i) if $P_0 < M$ then the denominator is a sum of two positive terms; if $P_0 = M$ the separation algorithm actually fails because you divided by 0 to get started but the formula actually recovers the constant equilibrium solution $P(t) \equiv M$; and if $P_0 > M$ then $|M - P_0| < P_0$ so the second term in the denominator can never be negative enough to cancel out the positive P_0 , for $t > 0$.)

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Question: You have a couple of homework problems where you are asked to find solutions $x(t)$ to differential equations of the form

$$x'(t) = a(x - b) \cdot (x - c).$$

How would you proceed?

<u>Application</u>

The Belgian demographer P.F. Verhulst introduced the logistic model around 1840, as a tool for studying human population growth. Our text demonstrates its superiority to the simple exponential growth model, and also illustrates why mathematical modelers must always exercise care, by comparing the two models to actual U.S. population data.

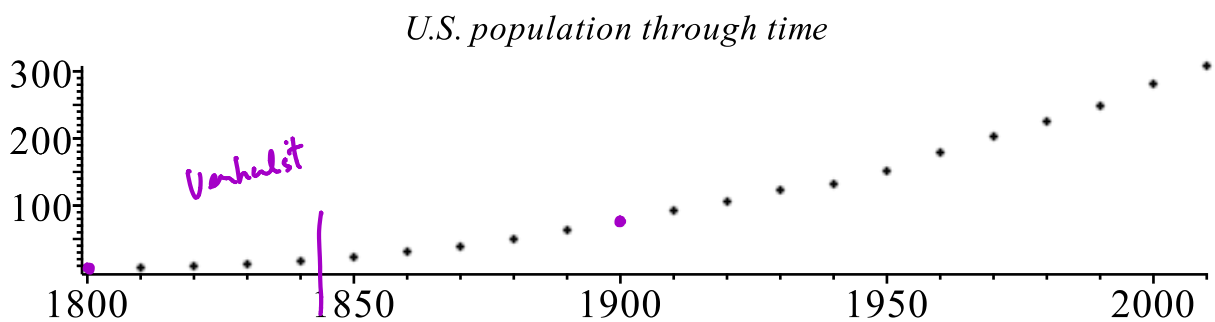
```
> restart : # clear memory
  Digits := 5 : #work with 5 significant digits
```

P_0

```
> pops := [[1800, 5.3], [1810, 7.2], [1820, 9.6], [1830, 12.9],
     [1840, 17.1], [1850, 23.2], [1860, 31.4], [1870, 38.6],
     [1880, 50.2], [1890, 63.0], [1900, 76.2], [1910, 92.2],
     [1920, 106.0], [1930, 123.2], [1940, 132.2], [1950, 151.3],
     [1960, 179.3], [1970, 203.3], [1980, 225.6], [1990, 248.7],
     [2000, 281.4], [2010, 308.]] : #I added 2010 - between 306-313
      # I used shift-enter to enter more than one line of information
      # before executing the command.
```

U.S. populations since 1800

```
> with(plots) : # plotting library of commands
  pointplot(pops, title = `U.S. population through time`);
```




```
>
```

Unlike Verhulst, the book uses data from 1800, 1850 and 1900 to get constants in our two models. We let t=0 correspond to 1800.

Exponential Model: For the exponential growth model $P(t) = P_0\, e^{r\,t}$ we use the 1800 and 1900 data to get values for P_0 and r :

need P_0, r

pop in 1900

```
> P0 := 5.308;
  solve(P0·exp(r·100) = 76.212, r);
```

$$P0 := 5.308$$

$$0.026643 \qquad (1)$$

```
> P1 := t→5.308· exp(.02664· t);#exponential model -eqtn (9) page 83
```

$$P1 := t \rightarrow 5.308\, e^{0.02664\, t} \qquad (2)$$

```
>
```

Logistic Model: We get P_0 from 1800, and use the 1850 and 1900 data to find k and M :

```
> P2 := t→M·P0/(P0 + (M−P0)·exp(−M·k·t)); # logistic solution we worked out
```

$$P2 := t \rightarrow \frac{M\, P0}{P0 + (M - P0)\, \mathrm{e}^{-M k t}} \tag{3}$$

→P_0, $P(100)$
$P(50)$

```
> solve({P2(50) = 23.192, P2(100) = 76.212}, {M, k});
```

$$\{M = 188.12,\ k = 0.00016772\} \tag{4}$$

```
> M := 188.12;
  k := .16772e-3;
  P2(t); #should be our logistic model function,
    #equation (11) page 84.
```

$$M := 188.12$$

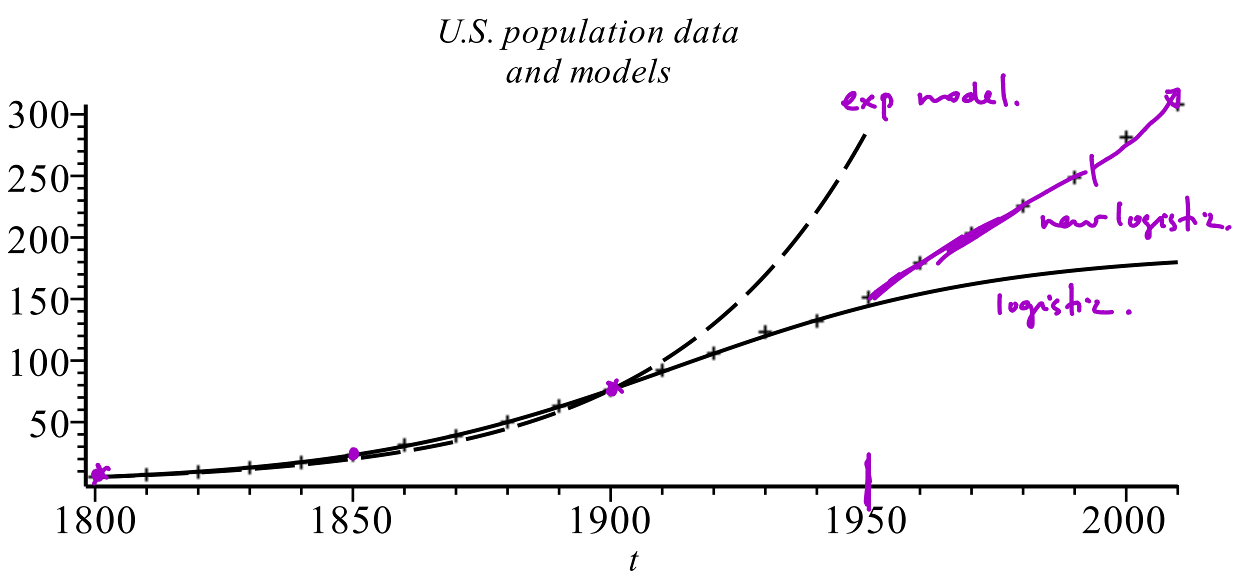
$$k := 0.00016772$$

$$\frac{998.54}{5.308 + 182.81\, \mathrm{e}^{-0.031551\, t}} \tag{5}$$

```
>
```

Now compare the two models with the real data, and discuss. The exponential model takes no account of the fact that the U.S. has only finite resources. Any ideas on why the logistic model begins to fail (with our parameters) around 1950?

```
> plot1 := plot(P1(t−1800), t = 1800..1950, color = black, linestyle = 3) :
   #this linestyle gives dashes for the exponential curve
  plot2 := plot(P2(t−1800), t = 1800..2010, color = black) :
  plot3 := pointplot(pops, symbol = cross) :
  display({plot1, plot2, plot3}, title = `U.S. population data
  and models`);
```




```
>
>
```

technology changed M.
life expectancies went up; δ went down
fertility rate went up.

Wed · finish logistic DE discussion Monday notes
· then start today's notes §2.2

Math 2280-01
Wed Jan ~~28~~ 25
2.2: Autonomous Differential Equations.

Recall, that a general first order DE for $x = x(t)$ is written in standard form as

$$x' = f(t, x) ,$$

which is shorthand for $x'(t) = f(t, x(t))$.
Definition: If the slope function f only depends on the value of $x(t)$, and not on t itself, then we call the first order differential equation *autonomous*:

$$x' = f(x) .$$

$P'(t) = kP(M-P)$

how fast x is changing only depends on value of x.

Example: The logistic DE, $P' = k\,P(M-P)$ is an autonomous differential equation for $P(t)$.

Definition: Constant solutions $x(t) \equiv c$ to autonomous differential equations $x' = f(x)$ are called *equilibrium solutions*. Since the derivative of a constant function $x(t) \equiv c$ is zero, the values c of equilibrium solutions are exactly the roots c to $f(c) = 0$.

const soln to $x'(t) = f(x)$
$x(t) \equiv c$: LHS$=0 \Rightarrow$ RHS$=0$
$f(c) = 0$

Example: The functions $P(t) \equiv 0$ and $P(t) \equiv M$ are the equilibrium solutions for the logistic DE.

Exercise 1: Find the equilibrium solutions of

1a) $x'(t) = 3x - x^2$ $= x(3-x)$

equil solns $x(t) \equiv 0$, $x(t) \equiv 3$.
equil pts $x = 0, 3$

1b) $x'(t) = x^3 + 2x^2 + x$ $= x(x^2+2x+1)$
$= x(x+1)^2$

equil sol's $x \equiv 0, -1$

1c) $x'(t) = \sin(x)$.

$x = 0, \pm\pi, \pm 2\pi, \ldots$
$x = k\pi$ $k \in \mathbb{Z}$

Def: Let $x(t) \equiv c$ be an equilibrium solution for an autonomous DE. Then

start close

stay close.

· c is a *stable* equilibrium solution if solutions with initial values close enough to c stay close to c. There is a precise way to say this, but it requires quantifiers: For every $\varepsilon > 0$ there exists a $\delta > 0$ so that for solutions with $|x(0) - c| < \delta$, we have $|x(t) - c| < \varepsilon$ for all $t > 0$.

· c is an *unstable* equilibrium if it is not stable.

· c is an *asymptotically stable* equilibrium solution if it's stable and in addition, if $x(0)$ is close enough to c, then $\lim_{t \to \infty} x(t) = c$, i.e. there exists a $\delta > 0$ so that if $|x(0) - c| < \delta$ then $\lim_{t \to \infty} x(t) = c$. (Notice that this means the horizontal line $x = c$ will be an *asymptote* to the solution graphs $x = x(t)$ in these cases.)

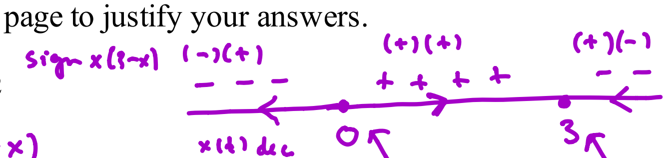
note if slope fcn $f(x)$ is cont, & $\frac{\partial f}{\partial x}$ is cont. then by uniqueness theorem no two solution graphs can even touch.

$x'(t) = f(x)$

Exercise 2: Use phase diagram analysis to guess the stability of the equilibrium solutions in Exercise 1. For (a) you've worked out a solution formula already, so you'll know you're right. For (b), (c), use the Theorem on the next page to justify your answers.

2a) $x'(t) = 3x - x^2$

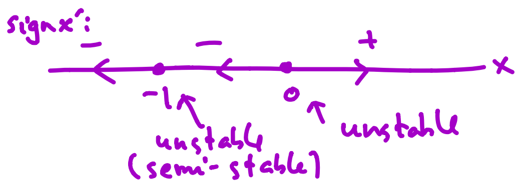
$= x(3-x)$



phase diagram

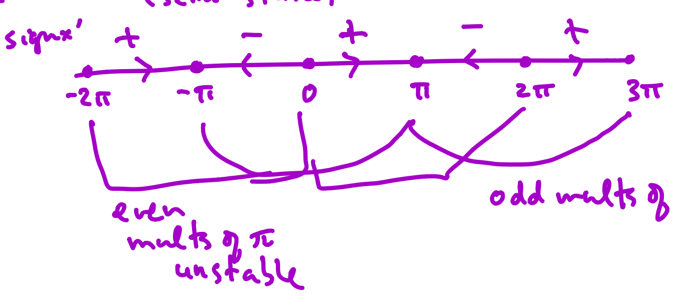
2b) $x'(t) = x^3 + 2x^2 + x$

$= x(x+1)^2$



2c) $x'(t) = \sin(x)$.

$= 0$ @ $x = k\pi$



asymptotically stable

Theorem: Consider the autonomous differential equation

$$x'(t) = f(x) .$$

with $f(x)$ and $\frac{\partial}{\partial x} f(x)$ continuous (so local existence and uniqueness theorems hold). Let $f(c) = 0$, i.e. $x(t) \equiv c$ is an equilibrium solution. Suppose c is an *isolated zero* of f, i.e. there is an open interval containing c so that c is the only zero of f in that interval. The the stability of the equilibrium solution c can is completely determined by the local phase diagrams:

$$sign(f): \; ----\underset{c}{0}+++ \quad \Rightarrow \quad \leftarrow\leftarrow\leftarrow c \rightarrow\rightarrow\rightarrow \quad \Rightarrow \quad c \text{ is unstable}$$

$$sign(f): \; +++0---- \quad \Rightarrow \quad \rightarrow\rightarrow\rightarrow c \leftarrow\leftarrow\leftarrow \quad \Rightarrow \quad c \text{ is asymptotically stable}$$

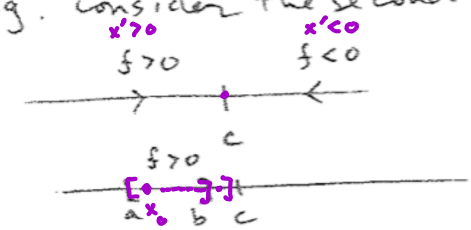
$$sign(f): \; +++0+++ \quad \Rightarrow \quad \rightarrow\rightarrow\rightarrow c \rightarrow\rightarrow\rightarrow \quad \Rightarrow \quad c \text{ is unstable (half stable)}$$

$$sign(f): ----0---- \quad \Rightarrow \quad \leftarrow\leftarrow\leftarrow c \leftarrow\leftarrow\leftarrow \quad \Rightarrow \quad c \text{ is unstable (half stable)}$$

You can actually prove this Theorem with calculus!! (want to try?)

Here's why!

e.g. consider the second case



f cont; f > 0 on subinterval [a, b]

$\Rightarrow f \geq \delta > 0$ on $[a, b]$

$x'(t) = f(x) \geq \delta$ as long as $x \in [a, b]$

(extreme value thm from calculus, f attains its minimum)

$\Rightarrow x'(t) \geq \delta$ as long as $x(t) \in [a, b]$

$\Rightarrow x(t)$ stays in this interval for time interval at most $\frac{b-a}{\delta}$

speed · time = dist

time = $\frac{\text{dist}}{\text{speed}}$.

$\Rightarrow \lim_{t \to \infty} x(t) = c$

(because we can pick the right endpoint b as close to c (b < c) as we want, to ensure x(t) eventually gets as close as we want to the value c)

Exercise 3) Use the chain rule to check that if $x(t)$ solves the autonomous DE

$$x'(t) = f(x) : \; x'(\tau) = f(x(\tau))$$

Then $\boxed{X(t) := x(t-a)}$ solves the same DE. What does this say about the geometry of representative solution graphs to autonomous DEs? Have we already noticed this?

Start here Friday: Check: LHS: $X'(t) = x'(t-a) \cdot 1 = f(x(t-a)) =$

$= f(X(t))$

Further application: Doomsday-extinction. With different hypotheses about fertility and mortality rates, one can arrive at a population model which looks like logistic, except the right hand side is the opposite of what it was in that case:

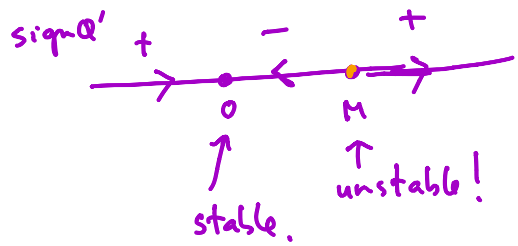
Logistic: $\quad P'(t) = -a\,P^2 + b\,P \qquad kP(M-P)$

Doomsday-extinction: $\quad Q'(t) = a\,Q^2 - b\,Q$

For example, suppose that the chances of procreation are proportional to population density (think alligators or crickets), i.e. the fertility rate $\beta = a\,Q(t)$, where $Q(t)$ is the population at time t. Suppose the morbidity rate is constant, $\delta = b$. With these assumptions the birth and death rates are $a\,Q^2$ and $-b\,Q$ .... which yields the DE above. In this case factor the right side:

$$Q'(t) = a\,Q\left(Q - \frac{b}{a}\right) = k\,Q(Q - M). \qquad kQ(Q-M)$$

Exercise 4a) Construct the phase diagram for the general doomsday-extinction model and discuss the stability of the equilbrium solutions.



[$Q_0 > M$ "doomsday" in finite time

$Q_0 < M$ $Q(t) \to 0$ as $t \to \infty$

"extinction"

Exercise 4b) If $P(t)$ solves the logistic differential equation

$$P'(t) = k\,P(M-P)$$ opposites

show that $Q(t) := P(-t)$ solves the doomsday-extinction differential equation

$$Q'(t) = k\,Q(Q-M)\,.$$

Use this to recover a formula for solutions to doomsday-extinction IVPs. What does this say about how representative solution graphs are related, for the logistic and the doomsday-extinction models? Recall, the solution to the logistic IVP is

$$P(t) = \frac{MP_0}{(M-P_0)e^{-Mkt} + P_0}\,.$$

IVP $\begin{cases} Q'(t) = kQ(Q-M) \\ Q(0) = Q_0 \end{cases}$

$$Q(t) = \frac{MQ_0}{(M-Q_0)e^{Mkt} + Q_0}$$

$Q_0 > M$: $|M-Q_0| < Q_0$

$$\boxed{\begin{aligned} &\text{if } x(t) \text{ solves} \\ &\quad x' = f(x) \\ &\text{then } z(t) = x(-t) \\ &\text{solves} \quad z' = -f(z) \\ &\text{Check:} \\ &z'(t) = x'(-t)\cdot(-1) \\ &\quad = -x'(-t) \\ &\quad = -f(x(-t)) \\ &\quad = -f(z(t))! \end{aligned}}$$

Exercise 5: Use your formula from the previous exercise or work the separable DE from scratch, to transcribe the solution to the doomsday-extinction IVP

$$x'(t) = x(x-1)$$
$$x(0) = 2\,.$$

$M = 1$
$Q_0 = 2$

Does the solution exist for all $t > 0$? (Hint: no, there is a very bad doomsday at $t = \ln 2$.)

$$x(t) = \frac{2}{-e^{t} + 2}\,.$$

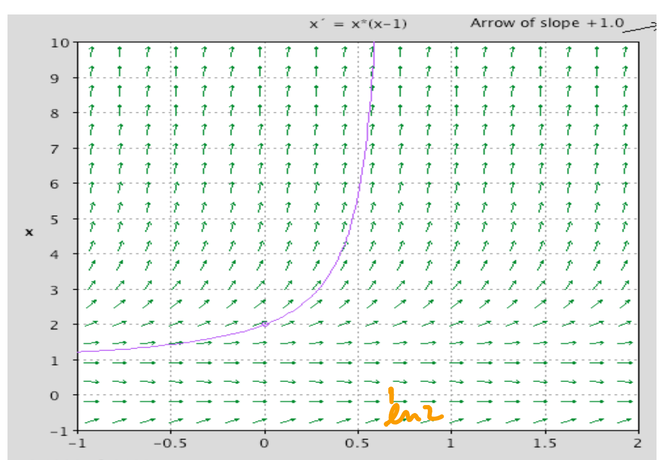
$$x(0) = \frac{2}{-1+2} = 2 \checkmark$$

vertical asymptote

$@\ -e^{t} + 2 = 0$

$2 = e^{t}$

$\ln 2 = t$



Math 2280-001
Fri Jan 30
2.2 Autonomous differential equations, with applications; 2.3 improved velocity models

Friday: • finish Wed notes
• do §2.2 part of today's

2.3 improved velocity: Monday.

- Recall that on Wednesday we discussed the following important concepts:
  - * Autonomous first order DE
  - * equilibrium solutions for autonomous DE's
  - * stability at equilibrium points.

Further application: (related to parts of a "yeast bioreactor" homework problem for next week) harvesting a logistic population...text p.89-91 (or, why do fisheries sometimes seem to die out "suddenly"?)
Consider the DE

$$\underbrace{P'(t) = a\,P - b\,P^2}_{\text{logistic}}\ \underbrace{-\,h}_{\text{harvesting term.}}\ . \quad = 2P - P^2 - h$$

Notice that the first two terms represent a logistic rate of change, but we are now harvesting the population at a rate of h units per time. For simplicity we'll assume we're harvesting fish per year (or thousands of fish per year etc.) One could model different situations, e.g. constant "effort" harvesting, in which the effect on how fast the population was changing could be $h\,P$ instead of P.

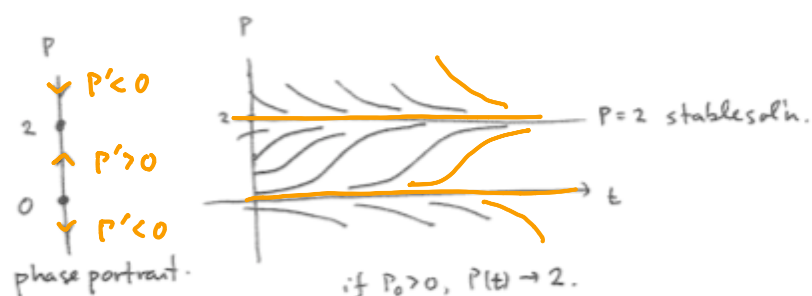
For computational ease we will assume $a = 2$, $b = 1$. (One could actually change units of population and time to reduce to this case.)

for computational simplicity take $a=2$, $b=1$

Case 0 no harvesting

standard logistic

$$P'(t) = 2P - P^2 = P(2-P)$$



if $P_0 > 0$, $P(t) \to 2$.
(units could be millions of fish, e.g.)

with harvesting:

$$P'(t) = 2P - P^2 - h$$

factor out -1

$$= -(P^2 - 2P + h)$$
$$= -(P-P_1)(P-P_2)$$
$$P_1, P_2 = \frac{2 \pm \sqrt{4-4h}}{2}$$
$$= 1 \pm \sqrt{1-h}$$

Case 1: subcritical harvesting

$0 < h < 1$

Case 2: Critical harvesting

$h = 1$

$P'(t) = -(P-1)^2$

$= -(P^2 - 2P + 1)$

Case 3: Overharvesting

$h > 1$

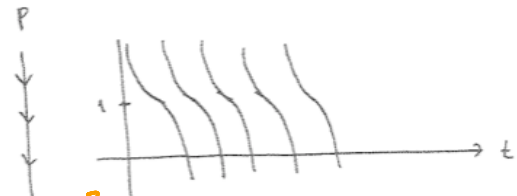
complex roots.

$P'(t) = -(P^2 - 2P + h)$

$= -[(P-1)^2 + (h-1)]$

$< 0.$

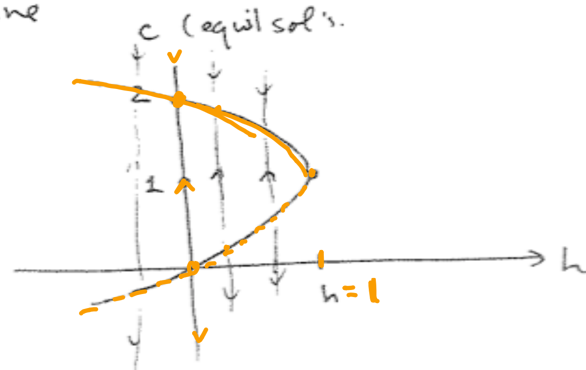
$P^2 - 2P + 1 + (h-1) = P^2 - 2P + h$ ✓



This model gives a plausible explanation for why many fisheries have "unexpectedly" collapsed in modern history. If $h < 1$ but near 1 and something perturbs the system a little bit (a bad winter, or a slight increase in fishing pressure), then the population and/or model could suddenly shift so that $P(t) \to 0$ very quickly.

Here's one picture that summarizes all the cases - you can think of it as collection of the phase diagrams for different fishing pressures h. The upper half of the parabola represents the stable equilibria, and the lower half represents the unstable equilibria. Diagrams like this are called "bifurcation diagrams". In the sketch below, the point on the h- axis should be labeled $h = 1$, not h. What's shown is the parabola of equilibrium solutions, $c = 1 \pm \sqrt{1 - h}$, i.e. $2c - c^2 - h = 0$, i.e. $h = c(2 - c)$.

"bifurcation diagram" of equilibrium solutions in the h-c plane



graph of equil solt'ns as a fcn of h.

roots $P = c$ of

$P^2 - 2P + h = 0$

$c = 1 \pm \sqrt{1-h}.$

$h = -P^2 + 2P$

Math 2280-001   Quiz 2   January 20, 2017
SOLUTIONS

1) Consider the initial value problem

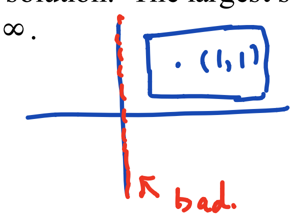
$$y'(x) = \frac{y^2}{x}$$
$$y(1) = 1.$$

initial point $(x_0, y_0) = (1,1)$

a) *Use the existence-uniqueness theorem to show that there some open interval containing $x_0 = 1$ on which this initial value problem has a unique solution. (Hint: Recall, if the slope function $f(x, y)$ is continuous in a coordinate rectangle R having the initial point in its interior, then there exists at least one solution. If the partial derivative $\frac{\partial}{\partial y} f(x, y)$ is also continuous, then the solution is unique as long as its graph remains inside R.)*

(2 points)

solution: The slope function $\boxed{f(x, y) = \frac{y^2}{x}}$ is continuous except along the $y-axis$, i.e. $x = 0$. The same holds true for $\frac{\partial}{\partial y}\left(\frac{y^2}{x}\right) = \boxed{\frac{2y}{x}}$. So any coordinate rectangle that contains the initial point $(1, 1)$ and avoids the $x - axis$ suffices to show that the IVP has a unique solution. The largest such coordinate rectangle is the "right half plane", given by $x > 0$, $-\infty < y < \infty$.



b) *The differential equation in this problem is separable, so you can actually find a solution to the initial value problem above. Do so.*

(6 points)

$$\frac{dy}{dx} = \frac{y^2}{x} \quad (x \neq 0)$$
$$\frac{dy}{y^2} = \frac{1}{x}\, dx \quad (y \neq 0)$$
$$\int y^{-2} dy = \int \frac{1}{x}\, dx$$
$$-y^{-1} = \ln|x| + C$$
$$y(1) = 1 \Rightarrow -1 = 0 + C \Rightarrow \underline{C = -1}$$
$$-\frac{1}{y} = \ln|x| - 1 = \ln(x) - 1 \ \text{ since } x > 0$$
$$y = -\frac{1}{\ln(x) - 1} = \frac{1}{1 - \ln x}$$

y undefined @ $x = e$
because $1 - \ln x = 0$ there.
graph has vert. asymp. @ $x = e$.
graph vertical @ $x = 0$
(DE doesn't make sense there)

c) *What is the largest x - interval on which the solution to* **b** *is defined as a differentiable function? Explain.*

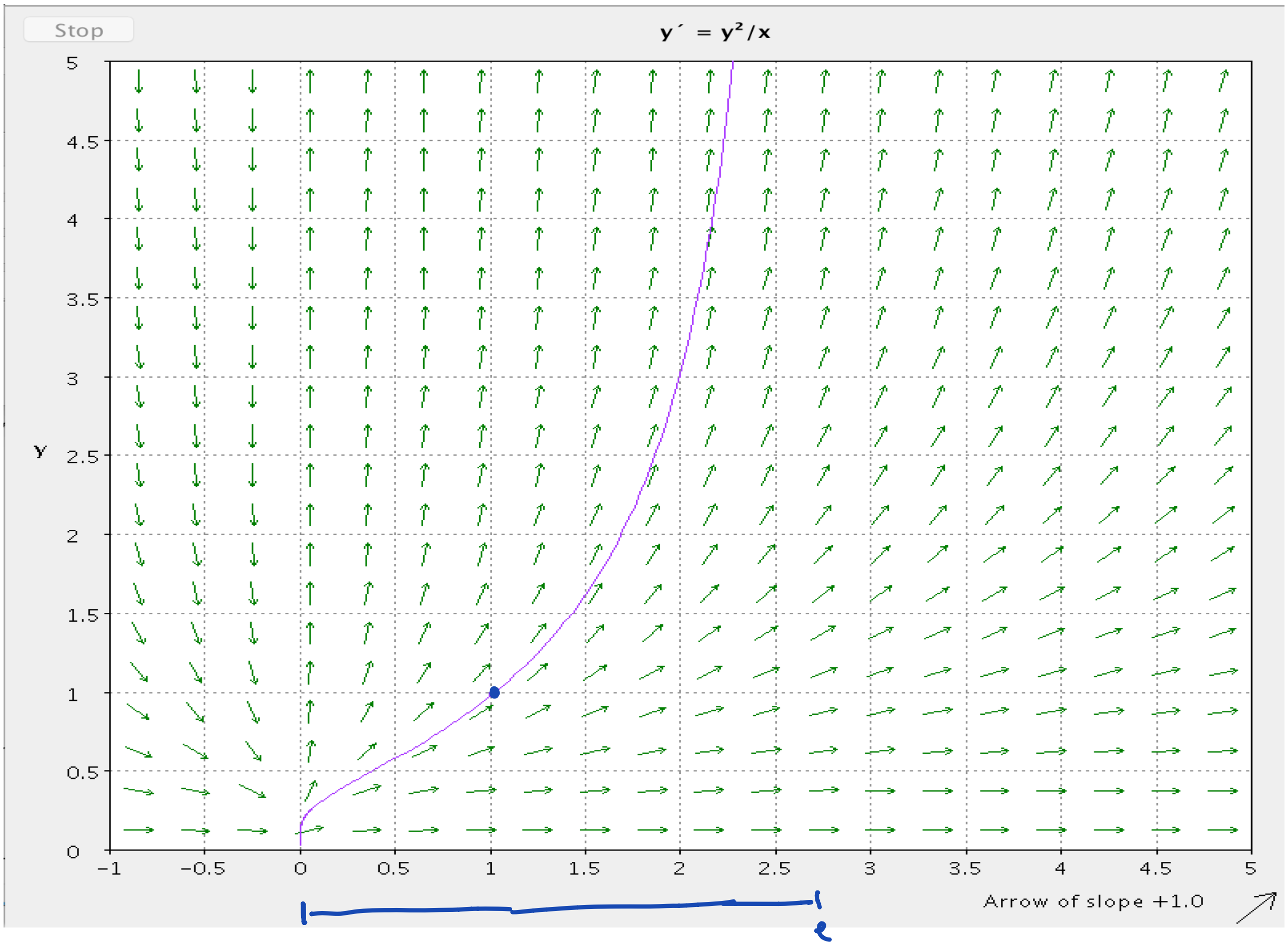
to IVP.

interval must contain $x_0 = 1$

(2 points)

solution: The largest interval containing $x_0 = 1$ on which $y(x)$ is not differentiable at $x = 0$, where y' blows up. ($\lim_{x \to 0^+} y'(x) = \lim_{x \to 0^+} \frac{y(x)^2}{x} = +\infty$). And, the graph has a vertical asymptote at $x = e$, where the denominator of $y(x)$ is zero. So the largest interval containing the initial point $x = 1$ on which the IVP solution exists is $0 < x < e$.

$(0, e)$


Stop
y´ = y²/x
y
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
−1
−0.5
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
Arrow of slope +1.0