on Friday, we discovered that IVP's might have more than one soltn.

Math 2280-001
Week 2, Jan 17-20; sections 1.3-1.4, part of 1.5
Wed Jan 18
1.3-1.4 Existence-uniqueness theorem, separable differential equations, singular solutions, applications.

$\exists$!

- Go over the existence-uniqueness theorem from the end of class last Friday. (Use last Friday's notes.)
- There should be time at the end of class for us to try a few of your homework problems, so please look at these before Wednesday to decide which ones might interest you.

<u>Exercise 1</u> (A slight variation on Exercise 4 in Friday's notes. Also, one of your homework problems is similar.) Consider the IVP

$$\frac{dy}{dx} = y^{\left(\frac{2}{3}\right)}$$
$$y(3) = 8.$$

← same DE as Fri:

sep vars soltns $\boxed{y = \frac{1}{27}(x+c)^3}$

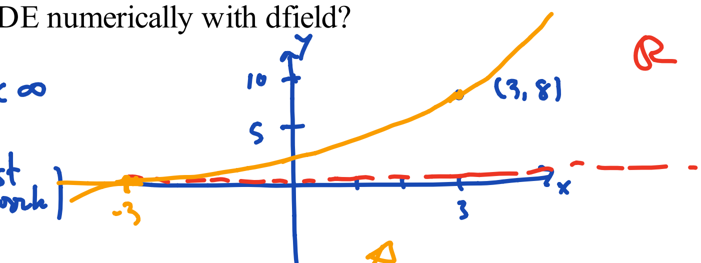
sing. soltn $y = 0$

<u>a)</u> Does the IVP above have a unique solution?
<u>b)</u> Find the IVP solution above, using separation of variables. What is the largest interval on which it is the unique solution?
<u>c)</u> What happens when you solve this DE numerically with dfield?

(a) choose $\mathcal{R}$: $-\infty < x < \infty$, $y > 0$

(that's the biggest one that will work)

(or $-2 < x < 4$, $1 < y < 10$)



slope fn $f(x,y) = y^{2/3}$ cont. in $\mathcal{R}$

$\frac{\partial f}{\partial y} = \frac{2}{3} y^{-1/3}$ cont. in $\mathcal{R}$ because x-axis is not in our rectangle.

so $\exists!$ soltn to IVP, for as long as graph stays in our rectangle.

(b) find actual unique soltn.

$y(3) = 8 = \frac{1}{27}(3+c)^3$

$2^3 = \frac{1}{3^3}(3+c)^3$

$2 = \frac{1}{3}(3+c) \Rightarrow c =$

$6 = 3 + c \Rightarrow c = 3$

$y(x) = \frac{1}{27}(x+3)^3$

solution unique for $x \geq -3$

<u>Exercise 2</u>: Do the initial value problems below always have unique solutions? Can you find them? (Notice two of these are NOT separable differential equations.) Can Maple find formulas for the solution functions?

<u>a)</u>

$$y' = (x+1)(y-3)^2$$
$$y(x_0) = y_0$$

$f(x,y)$ is cont, $\frac{\partial f}{\partial y}$ is cont $\}$ on all of $\mathbb{R}^2$

$\frac{dy}{dx} = (x+1)(y-3)^2$

- $\int \frac{dy}{(y-3)^2} = \int (x+1)\,dx \Longrightarrow \int (y-3)^{-2}\,dy = \int (x+1)\,dx$

$-(y-3)^{-1} = \frac{x^2}{2} + x + C$

could solve for y explicitly

also $y(x) \equiv 3$

$y' = 0$ LHS

$(x+1)(y-3)^2 = 0$ RHS

<u>b)</u>

$$y' = x^2 + y^2$$
$$y(x_0) = y_0$$

- $\exists!$ solns

slope functions & $\frac{\partial}{\partial y}$ of slope fcns are continuous on $\mathbb{R}^2$

<u>c)</u>

$$y' = x^4 + y^4$$
$$y(x_0) = y_0$$

- $\exists!$ solns

Exercise 2: Do the initial value problems below always have unique solutions? Can you find them? (Notice two of these are NOT separable differential equations.) Can Maple find formulas for the solution functions?

a)

$$y' = (x+1)(y-3)^2$$
$$y(x_0) = y_0$$

> $with(DEtools):$

> $dsolve\left(y'(x) = (x+1)\cdot(y(x)-3)^2, y(x)\right);$

$$y(x) = \frac{3x^2 + 6\_C1 + 6x - 2}{x^2 + 2\_C1 + 2x} \quad \textbf{(1)}$$

>

>

b)

$$y' = x^2 + y^2$$
$$y(x_0) = y_0$$

> $dsolve\left(y'(x) = x^2 + y(x)^2, y(x)\right);$

$$y(x) = \frac{\left(-\mathrm{BesselJ}\left(-\frac{3}{4}, \frac{1}{2}x^2\right)\_C1 - \mathrm{BesselY}\left(-\frac{3}{4}, \frac{1}{2}x^2\right)\right)x}{\_C1\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{1}{2}x^2\right) + \mathrm{BesselY}\left(\frac{1}{4}, \frac{1}{2}x^2\right)} \quad \textbf{(2)}$$

>

c)

$$y' = x^4 + y^4$$
$$y(x_0) = y_0$$

> $dsolve\left(y'(x) = x^4 + y(x)^4, y(x)\right);$

>

For your section 1.4 hw this week I assigned a selection of separable DE's - some applications will be familiar with from last week or previous courses, e.g. exponential growth and Newton's Law of cooling. Below is an application that might be new to you, and that illustrates conservation of energy as a tool for modeling differential equations in physics.

Toricelli's Law, for draining water tanks. Refer to the figure below.
Exercise 3:
a) Neglect friction, use conservation of energy, and assume the water still in the tank is moving with negligable velocity ($a << A$) . Equate the lost potential energy from the top in time dt to the gained kinetic energy in the water streaming out of the hole in the tank to deduce that the speed v with which the water exits the tank is given by

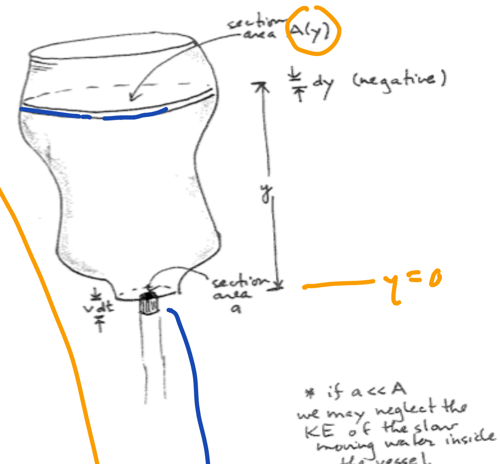
$$\boxed{v = \sqrt{2gy}}\ .$$

exit speed is faster when depth y is greater

when the water depth above the hole is $y(t)$ (and g is accel of gravity).

b) Use part (a) to derive the separable DE for water depth

$$\boxed{A(y)\frac{dy}{dt} = -k\sqrt{y}} \quad (k = a\sqrt{2g})\,.$$



a) consider time increment Δt (small).

est. $\Delta(T.E.)$

$TE = KE + PE$
total energy, kinetic ener., potential energy.

amt Δm of water mass leaves cistern.

$\Delta(TE) = \Delta(KE) + \Delta(PE)$

$0 \cong \frac{1}{2}(\cancel{\Delta m})v^2 - (\cancel{\Delta m})gy$

in the limit: $0 = \frac{1}{2}v^2 - gy$

$2gy = v^2 \rightarrow \boxed{v = \sqrt{2gy}}$

b) DE relate exit speed v to $\frac{dy}{dt}$: volume lost from top = volume exited from bottom in time increment Δt

top: $\Delta V \approx A(-\Delta y)$

bottom: $\Delta V \approx a \cdot v \cdot \Delta t$

section area a
time Δt
speed v
$h = v(\Delta t)$

$-A(\Delta y) \approx a \cdot v \cdot \Delta t$

$\frac{\Delta y}{\Delta t} \approx \frac{a \cdot v}{-A}$

$\Rightarrow \boxed{\frac{dy}{dt} = \frac{-a}{A(y)}\sqrt{2gy}}$

Experiment fun! (We have to postpone this.) I've brought a leaky nalgene canteen so we can test the Toricelli model. For a cylindrical tank of height h as below, the cross-sectional area $A(y)$ is a constant A, so the Toricelli DE and IVP becomes

$$\frac{dy}{dt} = -k\, y^{\frac{1}{2}}$$
$$y(0) = \cancel{h}\ 1$$

for cylinder

(different k).

Exercise 2a) Solve the differential equation IVP, and IVP. Note that $y \geq 0$, and that $y = 0$ is a singular solution that separation of variables misses. We may choose our units of length so that $h = 1$ is the maximum water height in the tank. Show that in this case the solution to the IVP is given by

$$y(t) = \left(1 - \frac{k}{2}t\right)^2$$

note $\exists!$ theorem. fails along $y=0$

$$\int \frac{dy}{y^{1/2}} = \int -k\, dt \qquad (y \neq 0)\ (y>0)$$
$$\int y^{-1/2}\, dy$$
$$2y^{1/2} = -kt + C$$
$$y(0)=1:\ 2\cdot 1^{1/2} = 0 + C$$
$$2 = C$$
$$2y^{1/2} = -kt + 2$$
$$y^{1/2} = -\frac{k}{2}t + 1$$
$$\boxed{y = \left(-\frac{k}{2}t+1\right)^2}\ \checkmark$$

(until the tank runs empty).

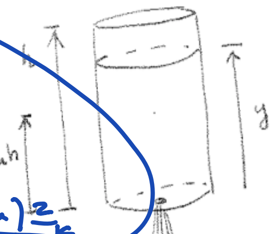
Exercise 2b: (We will use this calculation in our experiment) Setting the height $h = 1$ as in part 2a, let $T(\mu)$ be the time it takes the the water to go from height 1 (full) to height μ, where the fraction μ is between 0 and 1. Note, $T(1) = 0$ and $T(0)$ is the time it takes for the tank to empty completely. Show that $T(0)$ is related to $T(\mu)$ by

$$T(0)\left(1-\sqrt{\mu}\right) = T(\mu)\,, \quad \text{i.e.} \quad \boxed{T(0) = \frac{T(\mu)}{1-\sqrt{\mu}}}$$

$T(0)$: time it takes for tank to empty

$$0 = \left(-\frac{k}{2}t+1\right)^2$$
$$t = \frac{2}{k} = T(0)$$

$$T(\mu):\ \mu = \left(-\frac{k}{2}t+1\right)^2$$
$$\sqrt{\mu} = 1 - \frac{k}{2}t$$
$$\frac{k}{2}t = 1 - \sqrt{\mu} \Rightarrow t = \boxed{T(\mu) = (1-\sqrt{\mu})\frac{2}{k} = (1-\sqrt{\mu})\,T(0)}$$



Experiment! We'll time how long it takes to half-empty the canteen, and predict how long it will take to completely empty it when we rerun the experiment. Here are numbers I once got in my office, let's see how ours compare.

```
> Digits := 5 : # that should be enough significant digits
> 1/(1 - sqrt(.5)); # the factor from above, when mu is 0.5
                                        3.4143                      (1)
> Thalf := 35;   # seconds to half-empty canteen
  Tpredict := 3.4143·Thalf;  #prediction
                                   Thalf := 35
                                Tpredict := 119.50                  (2)
>
```

$$\mu = \tfrac{1}{2}$$
$$T(0) = \left(\frac{1}{1-\sqrt{\tfrac{1}{2}}}\right) T\left(\tfrac{1}{2}\right)$$