Exercise 4a) Use separation of variables to solve the IVP

$$\frac{dy}{dx} = y^{\left(\frac{2}{3}\right)}$$

$$\boxed{y(0) = 0}$$

4b) But there are actually a lot more solutions to this IVP! (Solutions which don't arise from the separation of variables algorithm are called singular solutions.) Once we find these solutions, we can figure out why separation of variables missed them.

4c) Sketch some of these singular solutions onto the slope field below.

4a) $\int \frac{dy}{y^{2/3}} = \int dx \leftarrow$ assumed $y^{2/3} \neq 0$ Case 1

$\int y^{-2/3}\, dy = \int dx$

$\boxed{3y^{1/3} = x + C}$

$\boxed{y^{1/3} = \frac{1}{3}x + C} \leftarrow$

IVP $y(0) = 0 \Rightarrow 0 = 0 + C$

$\Rightarrow C = 0$

$y^{1/3} = \frac{1}{3}x$

$y = \frac{x^3}{27}$

x	y
0	0
3	1
-3	-1

Case 1 get solns

$y = \left[\frac{1}{3}(x+C)\right]^3$

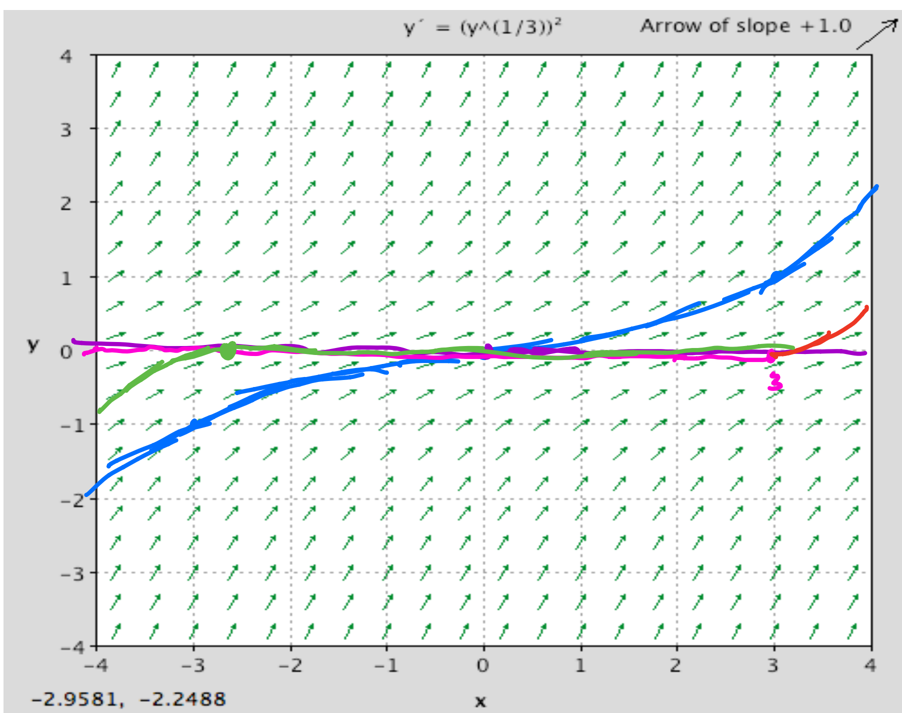
$y = \frac{1}{27}(x+C)^3$

Case 2 : $y = 0$

if $y(x) \equiv 0$, that's a soln

check $\Rightarrow y' = 0$ LHS

$(y(x))^{2/3} = 0$ RHS



3rd soln:

$$y(x) = \begin{cases} 0 & x \leq 3 \\ \frac{1}{27}(x-3)^3 & x \geq 0 \end{cases}$$

Here's what's going on (stated in 1.3 page 24 of text; partly proven in Appendix A.)

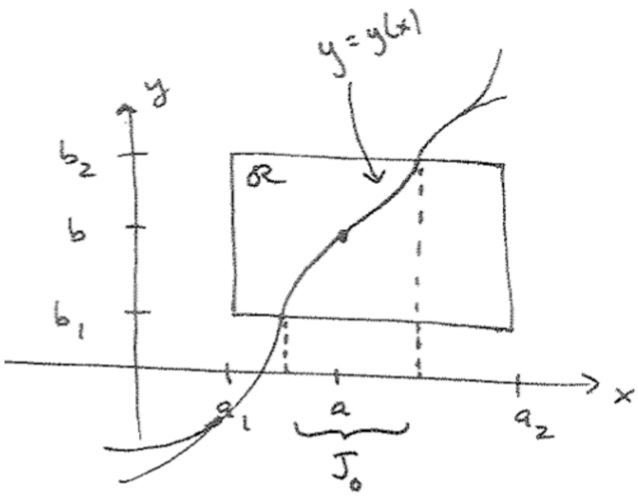
<u>Existence - uniqueness theorem for the initial value problem</u>

Consider the IVP

$$\frac{dy}{dx} = f(x, y)$$
$$y(a) = b$$

- Let the point (a, b) be interior to a coordinate rectangle $\mathcal{R} : a_1 \leq x \leq a_2, b_1 \leq y \leq b_2$ in the $x-y$ plane.
- <u>Existence:</u> If $f(x, y)$ is <u>continuous</u> in $\mathcal{R}$ (i.e. if two points in $\mathcal{R}$ are close enough, then the values of f at those two points are as close as we want). Then there exists a solution to the IVP, defined on some subinterval $J \subseteq [a_1, a_2]$.
- <u>Uniqueness:</u> If the partial derivative function $\frac{\partial}{\partial y} f(x, y)$ is also continuous in $\mathcal{R}$, then for any subinterval $a \in J_0 \subseteq J$ of x values for which the graph $y = y(x)$ lies in the rectangle, the solution is unique!

See figure below. The intuition for existence is that if the slope field $f(x, y)$ is continuous, one can follow it from the initial point to reconstruct the graph. The condition on the y-partial derivative of $f(x, y)$ turns out to prevent multiple graphs from being able to peel off.



<u>Exercise 5</u>: Discuss how the existence-uniqueness theorem is consistent with our work in Exercises 1-4 in today's notes, where we were able to find explicit solution formulas because the differential equations were actually separable (#1,3,4) or when the solution formula was given to us (#2).

on Friday, we discovered that IVP's might have more than one soltn.

Math 2280-001
Week 2, Jan 17-20; sections 1.3-1.4, part of 1.5
Wed Jan 18
1.3-1.4 Existence-uniqueness theorem, separable differential equations, singular solutions, applications.

$\exists$!

- Go over the existence-uniqueness theorem from the end of class last Friday. (Use last Friday's notes.)
- There should be time at the end of class for us to try a few of your homework problems, so please look at these before Wednesday to decide which ones might interest you.

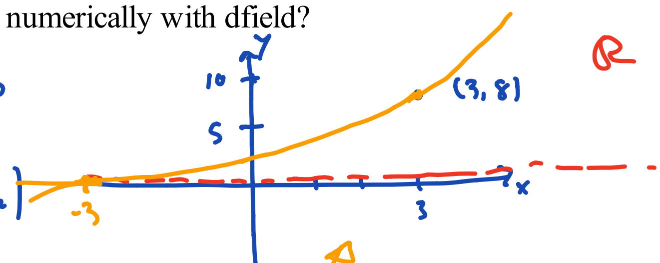
<u>Exercise 1</u> (A slight variation on Exercise 4 in Friday's notes. Also, one of your homework problems is similar.) Consider the IVP

$$\frac{dy}{dx} = y^{\left(\frac{2}{3}\right)} \quad \leftarrow \text{same DE as Fri}$$
$$y(3) = 8.$$

sep vars soltns $\boxed{y = \frac{1}{27}(x+c)^3}$
sing. soltn $y = 0$

<u>a)</u> Does the IVP above have a unique solution?
<u>b)</u> Find the IVP solution above, using separation of variables. What is the largest interval on which it is the unique solution?
<u>c)</u> What happens when you solve this DE numerically with dfield?

(a) choose $\mathcal{R}: -\infty < x < \infty$
$y > 0$
(that's the biggest one that will work)



(or $-2 < x < 4$
$1 < y < 10$)

slope fcn $f(x,y) = y^{2/3}$ cont. in $\mathcal{R}$
$\frac{\partial f}{\partial y} = \frac{2}{3} y^{-1/3}$ cont. in $\mathcal{R}$
because x-axis is not in our rectangle.

so $\exists !$ soltn to IVP, as long as graph stays in our rectangle.

(b) find actual unique soltn.

$y(3) = 8 = \frac{1}{27}(3+c)^3$
$2^3 = \frac{1}{3^3}(3+c)^3$
$2 = \frac{1}{3}(3+c) \Rightarrow c =$
$6 = 3 + c \Rightarrow c = 3$

$y(x) = \frac{1}{27}(x+3)^3$

solution unique for $x \geq -3$

Exercise 2: Do the initial value problems below always have unique solutions? Can you find them? (Notice two of these are NOT separable differential equations.) Can Maple find formulas for the solution functions?

a)

$$y' = (x+1)(y-3)^2$$
$$y(x_0) = y_0$$

$f(x,y)$ is cont, $\frac{\partial f}{\partial y}$ is cont } on all of $\mathbb{R}^2$

$$\frac{dy}{dx} = (x+1)(y-3)^2$$

• $\int \frac{dy}{(y-3)^2} = \int (x+1)\,dx \implies \int (y-3)^{-2}\,dy = \int (x+1)\,dx$

$$\boxed{-(y-3)^{-1} = \frac{x^2}{2} + x + C}$$

could solve for y explicitly

$\boxed{\text{also } y(x) \equiv 3}$

$y' = 0$ LHS

$(x+1)(y-3)^2 = 0$ RHS

b)

$$y' = x^2 + y^2$$
$$y(x_0) = y_0$$

• $\exists!$ soltns

slope functions & $\frac{\partial}{\partial y}$ of slope fcns are continuous on $\mathbb{R}^2$

c)

$$y' = x^4 + y^4$$
$$y(x_0) = y_0$$

• $\exists!$ soltns

For your section 1.4 hw this week I assigned a selection of separable DE's - some applications will be familiar with from last week or previous courses, e.g. exponential growth and Newton's Law of cooling. Below is an application that might be new to you, and that illustrates conservation of energy as a tool for modeling differential equations in physics.

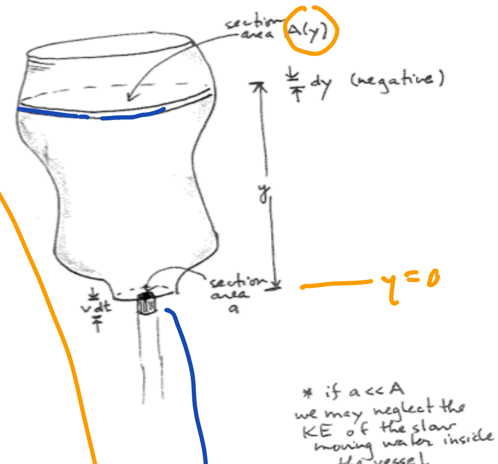
Toricelli's Law, for draining water tanks. Refer to the figure below.
Exercise 3:
a) Neglect friction, use conservation of energy, and assume the water still in the tank is moving with negligable velocity ($a << A$) . Equate the lost potential energy from the top in time *dt* to the gained kinetic energy in the water streaming out of the hole in the tank to deduce that the speed *v* with which the water exits the tank is given by

$$\boxed{v = \sqrt{2\,g\,y}}$$. exit speed is faster when depth y is greater

when the water depth above the hole is $y(t)$ (and *g* is accel of gravity).

b) Use part (a) to derive the separable DE for water depth

$$\boxed{A(y)\frac{dy}{dt} = -k\sqrt{y}} \quad (k = a\sqrt{2\,g}) .$$



a) consider time increment Δt (small).
est. Δ(T.E.)

TE = KE + PE
total energy, kinetic ener., potential energy.

amt Δm of water mass leaves cistern.

$\Delta(TE) = \Delta(KE) + \Delta(PE)$

$0 \approx \frac{1}{2}(\cancel{\Delta m})v^2 - (\cancel{\Delta m})gy$

in the limit: $0 = \frac{1}{2}v^2 - gy$

$2gy = v^2 \Rightarrow \boxed{v = \sqrt{2gy}}$

b) DE relate exit speed v to $\frac{dy}{dt}$: volume lost from top = volume exited from bottom in time increment Δt

section area a
time Δt
speed v
$h = v(\Delta t)$

top: $\Delta V \approx A(-\Delta y)$
bottom: $\Delta V \approx a \cdot v \cdot \Delta t$

$-A(\Delta y) \approx a \cdot v \cdot \Delta t$

$\frac{\Delta y}{\Delta t} \approx \frac{a \cdot v}{-A}$

$$\Rightarrow \boxed{\frac{dy}{dt} = \frac{-a}{A(y)}\sqrt{2gy}}$$

Experiment fun! (We have to postpone this.) I've brought a leaky nalgene canteen so we can test the Toricelli model. For a cylindrical tank of height h as below, the cross-sectional area $A(y)$ is a constant A, so the Toricelli DE and IVP becomes

for cylinder

$$\frac{dy}{dt} = -k\,y^{\frac{1}{2}}$$
$$y(0) = \cancel{h}\ 1$$

(different k).

Exercise 2a) Solve the differential equation IVP, and IVP. Note that $y \geq 0$, and that $y = 0$ is a singular solution that separation of variables misses. We may choose our units of length so that $h = 1$ is the maximum water height in the tank. Show that in this case the solution to the IVP is given by

$$y(t) = \left(1 - \frac{k}{2}t\right)^2$$

note ∃! theorem.
fails along $y=0$

$$\int \frac{dy}{y^{1/2}} = \int -k\,dt \qquad (y \neq 0)\ (y>0)$$
$$\int y^{-1/2}\,dy$$
$$2y^{1/2} = -kt + C$$
$$y(0)=1:\ 2\cdot 1^{1/2} = 0 + C$$
$$2 = C$$
$$2y^{1/2} = -kt + 2$$
$$y^{1/2} = -\frac{k}{2}t + 1$$
$$\boxed{y = \left(-\frac{k}{2}t + 1\right)^2}\ \checkmark$$

(until the tank runs empty).

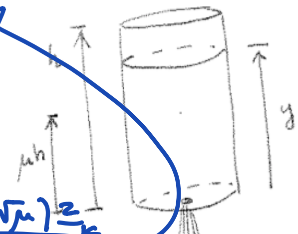
Exercise 2b: (We will use this calculation in our experiment) Setting the height $h = 1$ as in part 2a, let $T(\mu)$ be the time it takes the the water to go from height 1 (full) to height μ, where the fraction μ is between 0 and 1. Note, $T(1) = 0$ and $T(0)$ is the time it takes for the tank to empty completely. Show that $T(0)$ is related to $T(\mu)$ by

$$T(0)\left(1 - \sqrt{\mu}\right) = T(\mu), \quad \text{i.e.}\ \boxed{T(0) = \frac{T(\mu)}{1 - \sqrt{\mu}}}$$

$T(0)$: time it takes for tank to empty

$$0 = \left(-\frac{k}{2}t + 1\right)^2$$
$$t = \frac{2}{k} = T(0)$$

$$T(\mu):\ \mu = \left(-\frac{k}{2}t + 1\right)^2$$
$$\sqrt{\mu} = 1 - \frac{k}{2}t$$
$$\frac{k}{2}t = 1 - \sqrt{\mu} \Rightarrow t = \boxed{T(\mu)} = (1-\sqrt{\mu})\frac{2}{k} = \boxed{(1-\sqrt{\mu})\,T(0)}$$



Experiment! We'll time how long it takes to half-empty the canteen, and predict how long it will take to completely empty it when we rerun the experiment. Here are numbers I once got in my office, let's see how ours compare.

```
> Digits := 5 : # that should be enough significant digits
> 1/(1 - sqrt(.5)); # the factor from above, when mu is 0.5
                                    3.4143                              (1)
> Thalf := 35;   # seconds to half-empty canteen
  Tpredict := 3.4143 · Thalf;  #prediction
                                    Thalf := 35
                                 Tpredict := 119.50                     (2)
>
```

$\mu = 1/2$

$$T(0) = \left(\frac{1}{1-\sqrt{1/2}}\right) T(1/2)$$

Homework problems work space...we should have time to try at least a couple.

Math 2280-001
Fri Jan 23
(1.3-1.4 quiz is at the end of class)

$y' = f(x)g(y)$ Sep.

Section 1.5, linear differential equations:

A first order linear DE for $y(x)$ is one that can be (re)written as

$$y' + P(x)y = Q(x)$$

or for $x(t)$

$$x'(t) + P(t)x = Q(t)$$

Exercise 1: Classify the differential equations below as linear, separable, both, or neither. Justify your answers.

a) $y'(x) = -2y + 4x^2$

b) $y'(x) = x - y^2 + 1$

c) $y'(x) = x^2 - x^2 y + 1$

d) $y'(x) = \dfrac{6x - 3xy}{x^2 + 1}$

e) $y'(x) = x^2 + y^2$

f) $y'(x) = x^2 e^{x^3}$.

	linear	sep
a)	✓	no
b)	no	no
c)	✓	no
d)	✓	✓
e)	no	no
f)	✓	✓

$y' + \underbrace{2}_{P}y = \underbrace{4x^2}_{Q}$

$y' + x^2 y = x^2 + 1$

$y' + \dfrac{3x}{x^2+1} y = \dfrac{6x}{x^2+1}$

$y' = \dfrac{3x}{x^2+1}(2 - y)$

$y' + 0y = x^2 e^{x^3}$

$\dfrac{dy}{dx} = x^2 e^{x^3} \cdot 1$

(§1.2) $\quad g(y) = 1$

Recall that the method for solving separable differential equations via differentials was actually using the differentiation chain rule "backwards" to antidifferentiate and find the solution functions. The algorithm for solving linear DEs is a method to use the differentiation product rule backwards:

$$y' + P(x)y = Q(x)$$

Let $\int P(x)dx$ be any antiderivative of P. Multiply both sides of the DE by its exponential to yield an equivalent DE:

"integrating factor" I.F.

$$\boxed{e^{\int P(x)dx}(y' + P(x)y)} = e^{\int P(x)dx}Q(x)$$

This makes the left side a derivative (check via product rule):

$$\frac{d}{dx}(fg) = f'g + fg'$$

$$\frac{d}{dx}\left(\underbrace{e^{\int P(x)dx}}_{f}\ \underbrace{y}_{g}\right) = e^{\int P(x)dx}Q(x) .$$

$$\underbrace{e^{\int P(x)dx} \cdot P}_{f'}\ \underbrace{y}_{g} + e^{\int P(x)dx} \cdot y' = \text{LHS}$$

So you can antidifferentiate both sides with respect to x :

$$e^{\int P(x)dx}y = \int e^{\int P(x)dx}Q(x)dx \quad + C.$$

Dividing by the positive function $e^{\int P(x)dx}$ yields a formula for $y(x)$.

- Remark: If we abbreviate the function $e^{\int P(x)\,dx}$ by renaming it $G(x)$, then the formula for the solution $y(x)$ to the first order DE above is

$$y(x) = \frac{1}{G(x)}\int e^{\int P(x)\,dx} Q(x)\,dx + \frac{C}{G(x)}\ .$$

If x_0 is a point in any interval I for which the functions $P(x)$, $Q(x)$ are continuous, then $G(x)$ is positive and differentiable, and the formula for $y(x)$ yields a differentiable solution to the DE. By adjusting C to solve the IVP $y(x_0) = y_0$, we get a solution to the DE IVP on the entire interval. And, rewriting the DE as

$$y' = -P(x)y + Q(x)$$

we see that the existence-uniqueness theorem implies this is actually the only solution to the IVP on the interval (since $f(x, y)$ and $\frac{\partial}{\partial y} f(x, y) = P(x)$ are both continuous). These facts would not necessarily be true for separable DE's...and we've seen how separable DE solutions may not exist or be unique on arbitrarily large intervals.

Exercise 2: Solve the differential equation

$$y'(x) = -2\,y + 4\,x^2\ ,$$

and compare your solutions to the dfield plot below.

```
> with(DEtools) :    # load differential equations library
> deqtn2 := y'(x) = -2·y(x) + 4·x^2 :  #notice you must use · for multiplication in Maple,
                                        # and write y(x) rather than y.
  dsolve(deqtn2, y(x)); # Maple check
>
```

$$y' + 2y = 4x^2$$

$$P(x) = 2 \qquad Q(x) = 4x^2$$

$$\text{I.F. } e^{\int P(x)dx} = e^{2x}$$

$$e^{2x}(y' + 2y) = e^{2x}\,4x^2$$

$$\int: \quad \frac{d}{dx}(e^{2x}y) = e^{2x}\,4x^2$$

$$\underbrace{2e^{2x}y + e^{2x}y'}$$

$$e^{2x}y = \int \underbrace{4x^2}_{u}\,\underbrace{e^{2x}dx}_{dv}$$

$$du = 8x\,dx \qquad v = \frac{e^{2x}}{2}$$

$$= uv - \int v\,du$$

$$= 2x^2e^{2x} - \int \frac{e^{2x}}{2}\,8x\,dx$$

$$= 2x^2e^{2x} - \int \underbrace{4x}_{u}\,\underbrace{e^{2x}dx}_{dv}$$

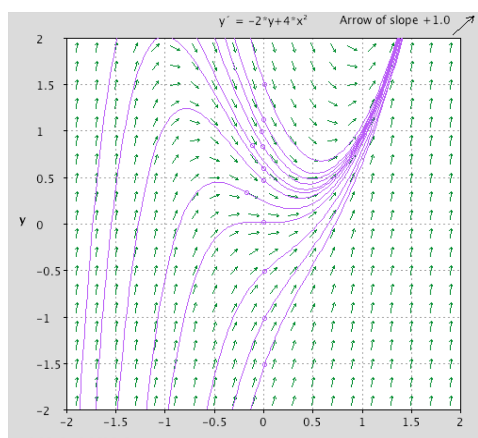
$$du = 4dx, \quad v = \frac{e^{2x}}{2}$$

$$= 2x^2e^{2x} - \Big[2xe^{2x} - \underbrace{\int 2e^{2x}dx}_{e^{2x}}\Big]$$

$$e^{2x}y = e^{2x}\left[2x^2 - 2x + 1\right] + C$$

$$\div e^{2x}$$

$$\boxed{y = 2x^2 - 2x + 1 + Ce^{-2x}}$$



Exercise 3: Find all solutions to the linear (and separable) DE

$$y'(x) = \frac{6x - 3xy}{x^2+1} \quad = \left(\frac{3x}{x^2+1}\right)(2-y)$$

Hint: as you can verify below, the general solution is $y(x) = 2 + C\left(x^2+1\right)^{-\frac{3}{2}}$.

> with(DEtools) :
> dsolve(

Separable : $\int \frac{dy}{y-2} = \int -\frac{3x}{x^2+1}\,dx$ ($y \neq 2$, $y(x) \equiv 2$ is soln)

$\ln|y-2| = -\frac{3}{2}\ln(x^2+1) + C_1$ $\qquad \frac{d}{dx}\frac{3}{2}\ln(x^2+1) = \frac{3}{2}\frac{1}{x^2+1}\cdot 2x$

$e \qquad e$

$|y-2| = e^{C_1}\left[(x^2+1)^{-3/2}\right]$ $\qquad \left[e^{\ln(x^2+1)}\right]^{-3/2}$

$y - 2 = C(x^2+1)^{-3/2}$ $\qquad C = \pm e^{C_1}$

$$\boxed{y = 2 + C(x^2+1)^{-3/2}}$$

linear : $y' + \underbrace{\frac{3x}{x^2+1}}_{P} y = \frac{6x}{x^2+1}$

I.F. : $\int \frac{3x}{x^2+1}\,dx = \frac{3}{2}\ln(x^2+1)$

$e^{\int P(x)dx} = e^{\frac{3}{2}\ln(x^2+1)} = (x^2+1)^{3/2}$

$\underbrace{(x^2+1)^{3/2}}_{f}\left[\underbrace{y'}_{g'} + \frac{3x}{x^2+1}\underbrace{y}_{g}\right] = (x^2+1)^{3/2}\frac{6x}{x^2+1}$ $\qquad f'$

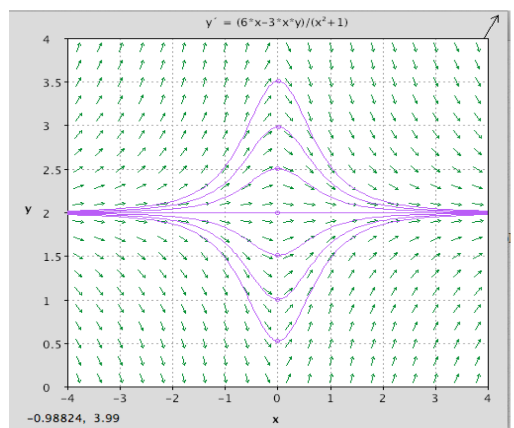
$\frac{d}{dx}\left[\underbrace{(x^2+1)^{3/2}}_{f}\underbrace{y}_{g}\right] = 6x(x^2+1)^{1/2}$

$f' = \frac{3}{\cancel{2}}(x^2+1)^{1/2}\cdot\cancel{2}x$

$\Rightarrow (x^2+1)^{3/2}y = \int 6x(x^2+1)^{1/2}\,dx$

$\frac{(x^2+1)^{3/2}y}{(x^2+1)^{3/2}} = \frac{2(x^2+1)^{3/2} + C}{(x^2+1)^{3/2}}$ $\qquad \frac{d}{dx}2(x^2+1)^{3/2} = 2\cdot\frac{3}{\cancel{2}}(x^2+1)^{1/2}\cancel{2}x$

$(fg)' = f'g + fg'$ $\qquad \div$ IF : $\boxed{y = 2 + C(x^2+1)^{-3/2}}$



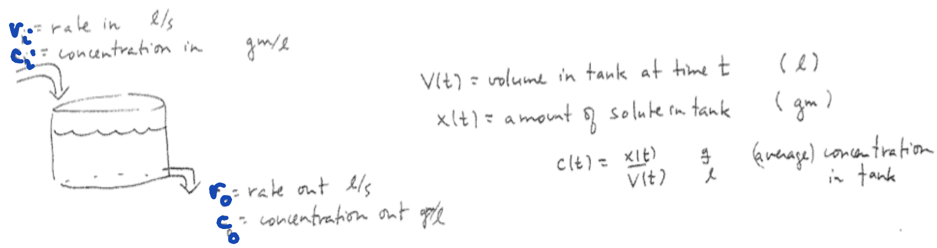
An extremely important class of modeling problems that lead to linear DE's involve input-output models. These have diverse applications ranging from bioengineering to environmental science. For example, The "tank" below could actually be a human body, a lake, or a pollution basin, in different applications.

For the present considerations, consider a tank holding liquid, with volume $V(t)$ (e.g units l). Liquid flows in at a rate r_i (e.g. units $\frac{l}{s}$ volume/time), and with solute concentration c_i (e.g. units $\frac{gm}{l}$ mass/volume). Liquid flows out at a rate r_o, and with concentration c_0. We are attempting to model the volume $V(t)$ of liquid and the amount of solute $x(t)$ (e.g. units gm mass) in the tank at time t, given $V(0) = V_0$, $x(0) = x_0$. We assume the solution in the tank is well-mixed, so that we can treat the concentration as uniform throughout the tank, i.e.

$$c_o = \frac{x(t)}{V(t)} \quad \frac{gm}{l}.$$

concentration everywhere in tank is the same i.e. average conc. $= \frac{x(t)}{V(t)}$ $\frac{\text{mass}}{\text{volume}}$.

See the diagram below.



Exercise 4: Under these assumptions use your modeling ability and Calculus to derive the following differential equations for $V(t)$ and $x(t)$:

a) The DE for $V(t)$, which we can just integrate:

$$\boxed{V'(t) = r_i - r_0}.$$

r_i: vol/time, r_0: vol/time

$$\int_0^t V'(\tau)\,d\tau = \int_0^t r_i(\tau) - r_o(\tau)\,d\tau \quad §1.2$$

so $V(t) = V_0 + \int_0^t r_i(\tau) - r_0(\tau)\,d\tau$

in time Δt

$\Delta V \approx \underbrace{r_i}_{\text{vol/time}} \underbrace{\Delta t}_{\text{time}} - r_o \Delta t$ (vol.)

$\frac{\Delta V}{\Delta t} \approx r_i - r_o$

$V'(t) = r_i - r_o$ vol/time

b) The linear DE for $x(t)$.

$$x'(t) = r_i c_i - r_o c_o = r_i c_i - r_o \frac{x}{V}.$$

$$\boxed{x'(t) + \frac{r_o}{V} x(t) = r_i c_i}$$

e.g. $r_i = 10 \frac{l}{min}$, $c_i = \frac{2g}{l}$ → solute coming into the tank $20 \frac{g}{min}$

$r_o = 5 \frac{l}{min}$, $c_o = \frac{x(t)}{V(t)} = 6 \frac{g}{l}$ → solute leaving $30 \frac{g}{min}$

Often (but not always) the tank volume remains constant, i.e. $r_i = r_o$. If the incoming concentration c_i is also constant, then the IVP for solute amount is

$$x' + a\,x = b$$
$$x(0) = x_0$$

where a, b are constants.

Exercise 5: The constant coefficient initial value problem above will recur throughout the course in various contexts, so let's solve it now. Hint: We will check our answer with Maple first, and see that the solution is

$$x(t) = \frac{b}{a} + \left(x_o - \frac{b}{a}\right)e^{-a\,t}.$$

Exercise 6 (taken from section 1.5 of text) Solve the following pollution problem IVP, to answer the follow-up question: Lake Huron has a relatively constant concentration for a certain pollutant. Since Lake Huron is the primary water source for Lake Erie, this is also the usual pollutant concentration in Lake Erie Due to an industrial accident, however, Lake Erie has suddenly obtained a concentration five times as large. Lake Erie has a volume of 480 km^3 , and water flows into and out of Lake Erie at a rate of 350 km^3 per year. Essentially all of the in-flow is from Lake Huron (see below). We expect that as time goes by, the water from Lake Huron will flush out Lake Erie. Assuming that the pollutant concentration is roughly the same everywhere in Lake Erie, about how long will it be until this concentration is only twice the original background concentration from Lake Huron?

http://www.enchantedlearning.com/usa/statesbw/greatlakesbw.GIF

a) Set up the initial value problem. Maybe use symbols c for the background concentration (in Huron),

$V = 480\ km^3$

$r = 350\ \dfrac{km^3}{y}$

b) Solve the IVP, and then answer the question.