Exercise 4) Newton's law of cooling is a model for how objects are heated or cooled by the temperature of an ambient medium surrounding them. In this model, the body temperature $T = T(t)$ changes at a rate proportional to the difference between it and the ambient temperature $A(t)$. In the simplest models A is constant.

a) Use this model to derive the differential equation

$$\frac{dT}{dt} = -k(T - A)\,.$$

a) $T'(t) = \tilde{k}(T-A)$

$T > A$ expect $T' < 0$, cooling

$T' < 0 = \tilde{k}(T-A) = \tilde{k}\cdot$positive $\Rightarrow \tilde{k} < 0$

(also, $T < A$ expect $T' > 0 \Rightarrow \tilde{k} < 0$)

So rewrite as in (a), with $\tilde{k} = -k$, so now $k > 0$ (just because we like positive constants)

b) Would the model have been correct if we wrote $\dfrac{dT}{dt} = k(T - A)$ instead? yes

c) Use this model to partially solve a murder mystery: At 3:00 p.m. a deceased body is found. Its temperature is 70 $^\circ$ F. An hour later the body temperature has decreased to 60 $^\circ$. It's been a winter inversion in SLC, with constant ambient temperature 30 $^\circ$. Assuming the Newton's law model, estimate the time of death.

ambient temp A

object $T(t)$

c) steps: solve DE:

$$\int \frac{1}{T-A}\,dT = \int -k\,dt$$

$$\ln|T-A| = -kt + C_1$$

$$|T-A| = e^{-kt}e^{C_1}$$

$$T - A = Ce^{-kt} \qquad C = e^{C_1} \text{ or } -e^{C_1}$$

$$\boxed{T = A + Ce^{-kt}}\,.$$

rest of steps: set $t = 0$ @ 3:00 pm.
measure time in hours

$A = 30$

$T(t) = 30 + Ce^{-kt}$

$T(0) = 70$
$T(1) = 60$
} use this info to find C & k, then set $T(t) = 98.6$ & solve for t.

$T(t) = 30 + 40e^{-kt}$

$T(1) = \cancel{30} + 40e^{-k\cdot 1} = \cancel{60}\ 30$

$e^{-k} = \frac{30}{40} = .75$

$\ln[$

$-k = \ln(.75) \Rightarrow k = -\ln(.75) = .2877$

$98.6 = 30 + 40e^{-kt}$

solve for t: $t = -1.84$ hours.

@ 1.16 hours

1:096 o'clock.

Math 2280-1
Wed January 11
HW due Friday ...
Quiz Friday ...

It works!

- Review from Monday. What were the main ideas we talked about?

What is a DE?
What is a soln to a DE?
What is an IVP (for a 1st order DE)
slope fields

- At the end of today's notes is the justification for "separation of variables". We will go over that at some point today.

Section 1.2: differential equations equivalent to ones of the form

$$y'(x) = f(x)$$

which we solve by direct antidifferentiation

$$y(x) = \int f(x)\, dx = F(x) + C.$$

Exercise 1 Solve the initial value problem

$$\frac{dy}{dx} = x\sqrt{x^2 + 4}$$
$$y(0) = 0$$

$$y = \int x\sqrt{x^2+4}\, dx = \frac{1}{3}(x^2+4)^{3/2} + C$$

$u = x^2+4$
$du = 2x\, dx$

$$\int u^{1/2} \tfrac{1}{2}\, du = \frac{1}{\cancel{2}} u^{3/2} \frac{\cancel{2}}{3} + C$$

shortcut:
ans is mult of $(x^2+4)^{3/2}$

$D_x \frac{1}{3}(x^2+4)^{3/2} = \frac{1}{3}\frac{3}{\cancel{2}}(x^2+4)^{1/2}\cancel{2}x$
$\frac{1}{\cancel{3}}\cancel{3}x(x^2+4)^{1/2}$

$$y(0) = 0 = \frac{4^{3/2}}{3} + C$$

$0 = \frac{8}{3} + C \Rightarrow C = -8/3$

$$\boxed{y(x) = \frac{1}{3}(x^2+4)^{3/2} - 8/3}$$

An important class of such problems arises in physics, usually as velocity/acceleration problems via Newton's second law. Recall that if a particle is moving along a number line and if $x(t)$ is the particle **position** function at time t, then the rate of change of $x(t)$ (with respect to t) namely $x'(t)$, is the **velocity** function. If we write $x'(t) = v(t)$ then the rate of change of velocity $v(t)$, namely $v'(t)$, is called the **acceleration** function $a(t)$, i.e.

$$x''(t) = v'(t) = a(t) .$$

$x(t) =$ position
$x'(t) = v(t)$ vel.
$v'(t) = a(t)$ accel.

Thus if $a(t)$ is known, e.g. from Newton's second law that force equals mass times acceleration, then one can antidifferentiate once to find velocity, and one more time to find position.

$$m\,x''(t) = f(t) \quad \text{net forces}$$
$$x''(t) = \tfrac{1}{m} f(t)$$

Exercise 2:
a) If the units for position are meters m and the units for time are seconds s, what are the units for velocity and acceleration? (These are *mks* units.)
b) Same question, if we use the English system in which length is measured in feet and time in seconds. Could you convert between *mks* units and English units?

$x(t)$ length

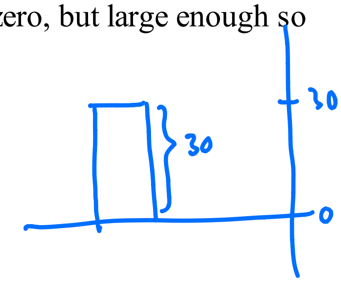
$v(t) = x'(t) = \lim\limits_{\Delta t \to 0} \dfrac{x(t+\Delta t) - x(t)}{\Delta t}$ $\quad \dfrac{\text{length}}{\text{time}}$

$a(t) = v'(t) = \lim\limits_{\Delta t \to 0} \dfrac{v(t+\Delta t) - v(t)}{\Delta t}$ $\quad \dfrac{\text{length/time}}{\text{time}} = \dfrac{\text{length}}{\text{time}^2}$

	mks	English
x	m	ft
v	m/s	ft/s
a	m/s²	ft/s²

Exercise 3: A projectile with very low air resistance is fired almost straight up from the roof of a building 30 meters high, with initial velocity 50 m/s. Its initial horizontal velocity is near zero, but large enough so that the object lands on the ground rather than the roof.

a) Neglecting friction, how high will the object get above ground?
b) When does the object land ?



$g = 9.8 \ m/s^2$

$m y'' = -mg$

$\int y'' = \int -g$

$v(t) = y'(t) = \int -g \, dt = -gt + C$.

@ $t=0$: $y'(0) = 0 + C$ so we call this v_0

$\int y'(t) = \int -gt + v_0 \, dt$

$y(t) = -g\frac{t^2}{2} + v_0 t + C$

@ $t=0$: $y(0) = 0 + 0 + C$ call this const y_0

$$\boxed{y(t) = -\frac{g}{2}t^2 + v_0 t + y_0}$$

for us: $\boxed{y(t) = -4.9t^2 + 50t + 30}$

$\boxed{v(t) = -9.8t + 50}$

a) at y_{max}, $v(t) = 0$

$-9.8t + 50 = 0$

$t = \frac{50}{9.8} = 5.097$

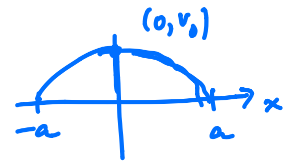
$y(t) = 157.42$ m.

b) Solve $y(t) = 0$ for t

10.77 sec.

Here's another fun example from section 1.2, which also reviews important ideas from Calculus - in particular we will see how the fact that the slope of a graph $y = g(x)$ is the derivative $\frac{dy}{dx}$ can lead to first order differential equations.

<u>Exercise 4</u>: (See "A swimmer's Problem" and Example 4 in section 1.2). A swimmer wishes to cross a river of width $w = 2\,a$, by swimming directly towards the opposite side, with constant transverse velocity v_S . The river velocity is fastest in the middle and is given by an even function of x , for $-a \le x \le a$. The velocity equal to zero at the river banks. For example, it could be that



river velocity

$$v_R(x) = v_0\left(1 - \frac{x^2}{a^2}\right).$$

See the configuration sketches below.

a) Writing the swimmer location at time t as $(x(t), y(t))$, translate the information above into expressions for $x'(t)$ and $y'(t)$.

b) The parametric curve describing the swimmer's location can also be expressed as the graph of a function $y = y(x)$. Show that $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = \frac{v_0}{v_S}\left(1 - \frac{x^2}{a^2}\right).$$

c) Compute an integral or solve a DE, to figure out how far downstream the swimmer will be when she reaches the far side of the river.

$$\boxed{\frac{4}{3}a\frac{v_0}{v_S}}$$

makes sense

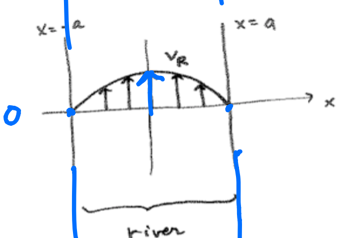
swimmer vel.

$x'(t) = v_S$

$y'(t) = v_R(x(t))$

$\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$

$y = 0$



tangent $\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}$

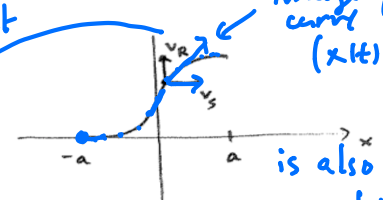


image of parametric curve $(x(t), y(t))$ is also graph $y = y(x)$

$\frac{d}{dt}y(x(t)) = \frac{dy}{dx}\frac{dx}{dt}$

$\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt}$

$\frac{dy/dt}{dx/dt}$

(b)

$$\boxed{\frac{dy}{dx} = \frac{v_R}{v_S} = \frac{v_0}{v_S}\left(1 - \frac{x^2}{a^2}\right)}$$

§1.2

(c): antidiff to find $y(x)$. With initial condition $y(-a) = 0$. Then compute $y(a)$.

(c'): shortcut: go $y(a) - y(-a)$ downriver $= \int_{-a}^{a} y'(x)\,dx$

$$= \int_{-a}^{a} \frac{v_0}{v_s}\underbrace{\left(1-\frac{x^2}{a^2}\right)}_{\text{even}} dx = 2\frac{v_0}{v_s}\left[x-\frac{x^3}{3a^2}\right]_0^a =$$

$$= 2\int_0^a \frac{v_0}{v_s}\left(1-\frac{x^2}{a^2}\right)dx = 2\frac{v_0}{v_s}\left[a-\frac{a^3}{3a^2}\right]$$

$$= 2\frac{v_0}{v_s}\frac{2}{3}a$$

$$\boxed{=\frac{4}{3}a\frac{v_0}{v_s}}$$

Exercise 5:

Suppose the acceleration function is a negative constant $-a$,

$$x''(t) = -a .$$

(This could happen for vertical motion, e.g. near the earth's surface with $a = g \approx 9.8\ \frac{m}{s^2} \approx 32\ \frac{ft}{s^2}$, as well as in other situations.)

a) Write $x(0) = x_0$, $v(0) = v_0$ for the initial position and velocity. Find formulas for $v(t)$ and $x(t)$.

b) Assuming $x(0) = 0$ and $v_0 > 0$, show that the maximum value of $x(t)$ is

$$x_{max} = \frac{1}{2}\frac{v_0^2}{a} .$$

(This formula may help with some homework problems.)

(skip)

1.4 Separable DE's: Important applications, as well as a lot of the examples we study in slope field discussions of section 1.3 are separable DE's. So let's discuss precisely what they are, and why the separation of variables algorithm works.

Definition: A separable first order DE for a function $y = y(x)$ is one that can be written in the form:

$$\boxed{\frac{dy}{dx} = f(x)\phi(y)}\,. \quad \leftarrow$$

It's more convenient to rewrite this DE as

$$\frac{1}{\phi(y)}\frac{dy}{dx} = f(x), \qquad \text{(as long as } \phi(y) \neq 0\text{)}\,.$$

Writing $g(y) = \dfrac{1}{\phi(y)}$ the differential equation reads

$$\boxed{g(y)\frac{dy}{dx} = f(x)}\,. \quad \leftarrow \text{integrate both } \cancel{\text{sides}} \text{ sides wrt } x$$

Solution (math justified): The left side of the modified differential equation is short for $g(y(x))\dfrac{dy}{dx}$. And if $G(y)$ is any antiderivative of $g(y)$, then we can rewrite this as

$$G'(y(x))y'(x)$$

which by the chain rule (read backwards) is nothing more than

$$\frac{d}{dx}G(y(x))\,.$$

And the solutions to

$$\frac{d}{dx}G(y(x)) = f(x)$$

are

$$\boxed{G(y(x)) = \int f(x)\,dx = F(x) + C.}$$

$$\int g(y(x))\frac{dy}{dx}\,dx = \int f(x)\,dx$$

$u = y(x)$

$du = y'(x)dx$

$\int g(u)\,du$

$G(u) + C_1$

$G(y(x)) + C_1 = F(x) + C_2$

where $F(x)$ is any antiderivative of $f(x)$. Thus solutions $y(x)$ to the original differential equation satisify

$$G(y) = F(x) + C\,. \quad \leftarrow \text{solns } y(x) \text{ solve this implicit eqtn}$$

This expresses solutions $y(x)$ implicitly as functions of x. You may be able to use algebra to solve this equation explicitly for $y = y(x)$, and (working the computation backwards) $y(x)$ will be a solution to the DE. (Even if you can't algebraically solve for $y(x)$, this still yields implicitly defined solutions.)

Solution (differential magic): Treat $\dfrac{dy}{dx}$ as a quotient of differentials dy, dx, and multiply and divide the DE to "separate" the variables:

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}\,.$$

$$\int g(y)dy = \int f(x)dx\,.$$

Antidifferentiate each side with respect to its variable (?!)

$$\int g(y)dy = \int f(x)dx\text{, i.e.}$$

$$G(y) + C_1 = F(x) + C_2 \Rightarrow \boxed{G(y) = F(x) + C\,.} \quad \text{Agrees!}$$

This is the same differential magic that you used for the "method of substitution" in antidifferentiation, which was essentially the "chain rule in reverse" for integration techniques.

Ex $\quad \dfrac{dy}{dx} = \dfrac{x}{y^2} \qquad \int y^2\,dy = \int x\,dx$

$\frac{1}{3}y^3 = \frac{1}{2}x^2 + C$, then if want to, solve explicitly for y