Math 2280-001
Week 9 concepts and homework Due Tuesday March 21
sections 4.1, 5.1-5.3

Postponed from last assignment:

4.1: modeling coupled mass-spring systems or multi-component input-output systems with systems of differential equations;converting single differential equations or systems of differential equations into equivalent first order systems of differential equations by introducing functions for the intermediate derivatives; comparing solutions to these equivalent systems.
4.1: 1, 3, **2**, 5, **8**, 17, 27, **30**, **31**, **32**.

week 8.1) This is related to problems17, 27 ideas above, and to our discussion in class that second order differential equations can be converted into equivalent first order systems of two differential equations.
a) Consider the IVP for the first order system of differential equations for $x(t), v(t)$:

$$x'(t) = v$$
$$v'(t) = -9\,x$$
$$x(0) = x_0.$$
$$v(0) = v_0 .$$

Find the equivalent second order differential equation initial value problem for the function $x(t)$. (Hint: $x(t)$ will represent simple harmonic motion in a certain undamped mass-spring problem.)
b) Use Chapter 3 techniques to solve the second order IVP for $x(t)$in part **a**.
c) Use your result from **b** to deduce the solutions $[x(t), v(t)]^T$ for the IVP in **a.**
d) Use your solution formulas for $x(t), v(t)$ from **c** along with algebra and trig identities to verify that the parametric solution curves $[x(t), v(t)]^T$ to the IVP in **a** lie on ellipses in the $x - v$ plane, satisfying implicit equations for ellipses given by

$$x^2 + \frac{v^2}{9} = C, \text{ where } C = x_0^2 + \frac{v_0^2}{9}.$$

e) Reproduce the result of **d** without using the solution formulas for $x(t), v(t)$, but instead by showing that whenever $x'(t) = v$ and $v'(t) = -9\,x$ then it must be true also that

$$\frac{d}{dt}\left(x(t)^2 + \frac{v(t)}{9}^2\right) \equiv 0\,,$$

so that $x(t)^2 + \frac{v(t)}{9}^2$ must be constant for any solution trajectory. Hint: use the chain rule to compute the time derivative above, then use the DE's in **a** to show that the terms cancel out.
f) What does your result in **e** have to do with conservation of energy for the undamped harmonic oscillator with mass $m = 1$ and spring constant $k = 9$? Hint: Recall that the total energy for a moving undamped mass-spring configuration with mass m and spring constant k is

$$TE = KE + PE = \frac{1}{2} m\, v^2 + \frac{1}{2} k\, x^2$$

g) Use pplane to create a picture of the tangent field for the first order system of differential equations in part a. Include selected solution trajectories. This "phase plane" picture should be consistent with your work above. Print out a screen shot to hand in.

New material:

5.1) recognizing homogeneous and non-homogeneous linear systems of first order differential equations; writing these systems in vector-matrix form; statement of existence and uniqueness for IVP's in first order systems of DE's and its consequences for the dimension of the solution space to the first order system, and for the general solution to the non-homogeneous system. Using the Wronskian to check for bases.
5.1: 11, **12**, **13**, 21, **22**, **31.**

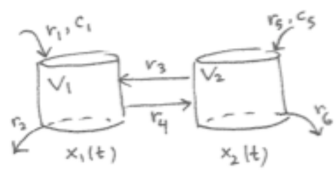
week 9.1) (lab) This is a continuation of **22**, **31**.
a) Use the eigenvalue-eigenvector ($e^{\lambda t}\underline{\mathbf{v}}$) method of section 5.2 to generate the basis $\{\underline{\mathbf{x}}_1(t), \underline{\mathbf{x}}_2(t)\}$ for the general solution that the text tells you in **22**.
b) Use pplane to draw the phase portrait for this first order system along with the parametric curve of the solution $[x(t), y(t)]^T$ to the initial value problem in **31**. Print out a screen shot of your work to hand in.

5.2) the eigenvalue-eigenvector method for finding the solution space to homogeneous constant coefficient first order systems of differential equations: real and complex eigenvalues.
5.2: 3, 13, 29, 31, **34**, 36. *In* **34** *you may use technology to find or check the eigendata.*

week 9.2) Use the eigenvalue-eigenvector method (with complex eigenvalues) to solve the first order system initial value problem which is equivalent to the second order differential equation IVP on the Friday March 3 notes that we discuss Monday. This is the reverse procedure from Monday, when we usedthe solutions from the equivalent second order DE IVP to deduce the solution to the first order system IVP. Of course, your answer here should be consistent with our work there.

$$\begin{bmatrix} x'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}$$
$$\begin{bmatrix} x(0) \\ v(0) \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$$

week 9.3 (lab) Consider a general input-output model with two compartments as indicated below. The compartments contain volumes V_1, V_2 and solute amounts $x_1(t), x_2(t)$ respectively. The flow rates (volume per time) are indicated by r_i, $i = 1..6$. The two input concentrations (solute amount per volume) are c_1, c_5.



a) What equalities between the flow rates guarantee that the volumes V_1, V_2 remain constant?
b) Assuming the equalities in **a** hold, what first order system of differential equations governs the rates of change for $x_1(t), x_2(t)$?

c) Suppose $r_2 = r_4 = r_6 = 100$, $r_3 = r_5 = 200$, $r_1 = 0 \ \frac{gal}{hour}$; $c_1 = 0$, $c_5 = 0.3 \ \frac{lb}{gal}$; $V_1 = V_2 = 100\ gal$.
Verify that the constant volumes are consistent with the rate balancing required in **a**. Then show that the general system in **b** reduces to the following system of DEs for the given parameter values:

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 60 \end{bmatrix}.$$

d) Solve the initial value problem for **c**, assuming there is initially no solute in either tank. Hint: Find the homogeneous solution; then find a particular solution which is a constant vector; and then use $\underline{\boldsymbol{x}} = \underline{\boldsymbol{x}}_P + \underline{\boldsymbol{x}}_H$ to solve the IVP.

e) Check your answer to **d** with technology, and hand in a copy of this verification. Wolfram Alpha should work well. In Maple, the "dsolve" command can solve systems of differential equations as well as single differential equations. For example, if I was checking my work in week 9.2 above, this is how I'd to it:

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> with(DEtools) :
> dsolve({x'(t) = v(t), v'(t) = -5·x(t) - 2·v(t), x(0) = 4, v(0) = -4});
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$$\{v(t) = -e^{-t}\,(4\cos(2\,t) + 8\sin(2\,t)),\ x(t) = 4\,e^{-t}\cos(2\,t)\} \qquad \textbf{(1)}$$

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Understanding phase portraits for first order homogeneous systems of two linear differential equations, $\underline{\mathbf{x}}' = A\,\underline{\mathbf{x}}$ *in terms of the eigendata and general solutions.*

5.3 **4**, **6**, **8**, **11**, 17-22.