Math 2280-001
Week 8 concepts and homework, due Tuesday March 7

Recall that all listed problems are good for seeing if you can work with the underlying concepts; that the underlined problems are to be handed in; and that the Friday quiz will be drawn from all of these concepts and from these or related problems.

3.6: *solving forced oscillation problems; understanding beating and resonance in undamped problems, steady periodic and transient solutions in damped problems, and practical resonance in slightly damped problems; mass-spring applications; finding natural frequencies in more general (undamped) conservative systems via conservation of energy equations.*
3.6: 3, 5, 7, 8, 11, 13, 17, 18, **20**, 21, **22** .

3.7: *The RLC circuit analog to mechanical vibrations.*
3.7: **11**

The following three extended problems are an excursion through the physical phenomena described in section 3.6:

Consider a mass-spring-dashpot system with additional external force $F(t)$ being applied to the mass. In particular, we consider a periodic external force $F(t) = F_0 \cos(\omega t)$. As we know, this system is governed by the following differential equation for the displacement $x(t)$ from equilibrium:

$$m x'' + c x' + k x = F_0 \cos(\omega t) .$$

We will take $m = 1\ kg$, $k = .64 \frac{N}{m}$, $F_0 = 2\ N$ in the following problems. Also, the system will always start at rest, i.e. $x(0) = 0$, $x'(0) = 0$. The damping coefficient c will be modified in different problems, as will the angular frequency ω of the driving force.

week 8.1) (lab) Consider the configuration above, in the undamped case $c = 0$. In particular consider the initial value problem

$$\begin{aligned} x'' + .64\, x &= 2 \cos(\omega t) \\ x(0) &= 0 \\ x'(0) &= 0 . \end{aligned}$$

a) What is the "natural" angular frequence ω_0 (for the unforced problem) in this differential equation? Hint, the natural frequency is defined to be the angular frequency for the solutions to the unforced and undamped differential equation, which in this case is the DE

$$x'' + .64\, x = 0.$$

b) Assume $\omega \neq \omega_0$: Use the method of undetermined coefficients to solve for the particular solution $x_P(t)$ for the forced differential equation. Then use $x(t) = x_P(t) + x_H(t)$ to solve the IVP. Check your answer with technology.

c) Write down the special case of the solution in **b** when $\omega = .7$. Compute the period of this solution, which is a superposition of two cosine functions. Use technology to graph one period of the solution. What phenomenon is exhibited by this solution?

d) Solve the IVP when $\omega = \omega_0$. Use the method of undetermined coefficients or operator factorization, i.e. $L = D^2 + .64\, I = [D + .8\, i\, I] \circ [D - .8\, i\, I]$ as in last week's homework, to find a particular solution, and then use $x = x_P + x_H$ to solve the IVP. Check your answer with technology. Graph the solution on the interval $0 \le t \le 60$ seconds. What phenomenon is exhibited by this solution?

week 8.2) (lab) Consider the same mass-spring dashpot system as above, except with $c = 2\ \frac{kg}{s}$, and with $\omega = 0.8$. This gives us the differential equation

$$x'' + 2\,x' + 0.64\,x = 2\cos(.8 \cdot t)$$

a) Use the method of undetermined coefficients to find a particular solution $x_P(t)$ to this differential equation.
b) Use your work from **a** and the solution to the corresponding homogeneous equation to write down the general solution to this differential equation. Identify the "steady periodic" and "transient" parts of this general solution.
c) Solve the initial value problem for this differential equation above, with $x(0) = 0, x'(0) = 0$. (You may use technology to do this, which will also check your work in parts a,b.)
d) Graph, on a single plot, the steady periodic solution from **b** and the solution to the initial value problem in **c**. Choose a time interval so that you can clearly see the convergence of the IVP solution to the steady periodic solution.

week 8.3) Consider the same forced oscillator equation, except with a relative small damping coefficient of $c = .2\ \frac{kg}{s}$. Unlike in problem week 8.2 we will consider variable angular frequencies ω. This gives the following differential equation

$$x'' + .2\,x' + 0.64\,x = 2\cos(\omega\, t)\ .$$

a) Use the formula (21) in section 3.6 of the text (which we also discuss in class) to create a plot of the amplitude C of the steady periodic solution $x_{sp}(t)$ as a function of the driving angular frequency ω.
b) Explain why the steady periodic amplitude peaks at a value near $\omega = 0.8$. Use calculus to find the exact value of ω which gives the maximum amplitude for the steady periodic solution.

Section 4.1 material below postponed until next homework assignment:

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4.1: modeling coupled mass-spring systems or multi-component input-output systems with systems of differential equations;converting single differential equations or systems of differential equations into equivalent first order systems of differential equations by introducing functions for the intermediate derivatives; comparing solutions to these equivalent systems.
4.1: 1, 3, **2**, 5, **8**, 17, 27, **30**, **31**, **32**.

week 8.4) This is related to problems17, 27 ideas above, and to our discussion in class that second order differential equations can be converted into equivalent first order systems of two differential equations.

a) Consider the IVP for the first order system of differential equations for $x(t)$, $v(t)$:

$$x'(t) = v$$
$$v'(t) = -9x$$
$$x(0) = x_0.$$
$$v(0) = v_0 .$$

Find the equivalent second order differential equation initial value problem for the function $x(t)$. (Hint: $x(t)$ will represent simple harmonic motion in a certain undamped mass-spring problem.)

b) Use Chapter 3 techniques to solve the second order IVP for $x(t)$ in part **a**.

c) Use your result from **b** to deduce the solutions $[x(t), v(t)]^T$ for the IVP in **a.**

d) Use your solution formulas for $x(t)$, $v(t)$ from **c** along with algebra and trig identities to verify that the parametric solution curves $[x(t), v(t)]^T$ to the IVP in **a** lie on ellipses in the $x - v$ plane, satisfying implicit equations for ellipses given by

$$x^2 + \frac{v^2}{9} = C, \text{ where } C = x_0^2 + \frac{v_0^2}{9}.$$

e) Reproduce the result of **d** without using the solution formulas for $x(t)$, $v(t)$, but instead by showing that whenever $x'(t) = v$ and $v'(t) = -9x$ then it must be true also that

$$\frac{d}{dt}\left(x(t)^2 + \frac{v(t)^2}{9}\right) \equiv 0,$$

so that $x(t)^2 + \frac{v(t)^2}{9}$ must be constant for any solution trajectory. Hint: use the chain rule to compute the time derivative above, then use the DE's in **a** to show that the terms cancel out.

f) What does your result in **e** have to do with conservation of energy for the undamped harmonic oscillator with mass $m = 1$ and spring constant $k = 9$? Hint: Recall that the total energy for a moving undamped mass-spring configuration with mass m and spring constant k is

$$TE = KE + PE = \frac{1}{2} m v^2 + \frac{1}{2} k x^2$$

g) Use pplane to create a picture of the tangent field for the first order system of differential equations in part a. Include selected solution trajectories. This "phase plane" picture should be consistent with your work above. Print out a screen shot to hand in.