Math 2280-001
Week 6 concepts and homework, due Friday February 17, at the start of class
(Our first midterm is also on Friday)

Recall that all problems are good for seeing if you can work with the underlying concepts; that the underlined problems are to be handed in; and that the Friday quiz will be drawn from all of these concepts and from these or related problems.

3.3: using the algorithm for finding the general solution to constant coefficient linear homogeneous differential equations: real roots, Euler's formula and complex roots, repeated roots; solving associated initial value problems.

3.3: 3, 9, 11, 23, 27.

carry over problems from last week:

w5.6) Do the following problems for homogeneous linear differential equations by hand. They are testing your ability to use the algorithm for finding bases for the solution spaces, based on the characteristic polynomial. Check your work with Maple (or other software). In Maple you will want to use the "dsolve" command to check differential equation solutions, and may want to use the "factor" command to check your factorizations of the characteristic polynomials. Hand in a printout of your computer verifications for the differential equation solutions, along with your written work.

a) Find the general solution to the differential equation for $y(x)$

$$y^{(3)} - 5\,y'' + 3\,y' + 9\,y = 0.$$

Hint: Find a root r_1 of the cubic characteristic polynomial, then divide it by $(r - r_1)$ to get a quotient quadratic polynomial.

b) Find the general solution to the differential equation for $x(t)$

$$x'' + 4\,x' + 13\,x = 0\,.$$

Hint: completing the square works well here - probably better than the quadratic formula.

c) Solve the initial value problem for the differential equation in b, with $x(0) = 0,\ x'(0) = 9$.

d) Find the general solution to the differential equation for $y(x)$

$$y^{(4)} - 8\,y' = 0.$$

e) Find the general solution to the differential equation for $y(x)$

$$y^{(5)} + 6\,y^{(3)} + 9\,y' = 0.$$

w5.7) Euler's formula

$$e^{i\,\theta} = \cos(\theta) + i\sin(\theta)$$

is extremely useful in higher mathematics/science/engineering. In class we discuss how this definition is motivated by Taylor series. Amazingly, the rule of exponents is true for such expressions. In other words

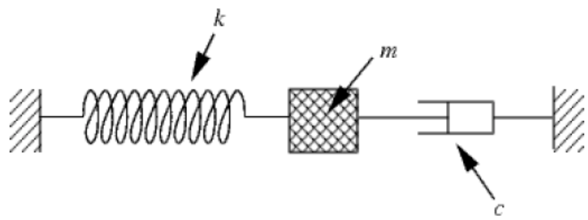
$$e^{i\,\alpha + i\,\beta} = e^{i\,\alpha}\, e^{i\,\beta}\,.$$

Check this identity by rewriting the left side as $e^{i\,(\alpha + \beta)}$ and then using Euler's formula to expand both sides. You'll notice that the identity is true because of the addition angle formulas for $\cos(\alpha + \beta)$ and $\sin(\alpha + \beta)$. (And so this gives a good way to recover the trig. identities if you happen to forget them.)

3.4: mechanical vibrations: setting up and solving differential equations and initial value problems for mechanical vibrations; underdamped, critically damped, and overdamped vibration problems; applications to pendulums, mass-spring configurations, and related problems; understanding and using phase-amplitude form for sinusoidal functions.

3.4: 3, **4**, 5, 6, **10**, (in **10** the weight of the buoy is $m\,g = \rho\,V g = \rho\,\pi\,r^2\,h\,g$. The text forgot to include the density ρ in computing the mass from the volume), 15, 17, 21.

w6.1 (lab) Consider the unforced mass-spring-dashpot system with $m = 3$ and $k = 48$. In the following problems the damping constant c will vary.



So, the displacement $x(t)$ of the mass from its equilibrium position will satsify the differential equation

$$3\,x''(t) + c\,x'(t) + 48\,x(t) = 0.$$

In each part below, we will consider the IVP with

$$x(0) = 3$$
$$x'(0) = -4.$$

a) Solve the IVP when there is no damping, i.e. $c = 0$. After finding your solution in linear combination form, convert into amplitude-phase form. Identify numerical values for amplitude, angular frequency, frequency, period, phase angle, and time delay.

b) Solve the IVP when $c = 1.2$ (we refer to the situation when c is small enough so that the characteristic polynomial ha complex roots as *underdamped)*. Instructions: When finding roots of characteristic polynomials, round off to 5 digits.

c) Recall that a value of the damping constant induces so-called *critical damping* whenever the associated characteristic polynomial has a double real root. Find the value of c that leads to critical damping in this problem, and solve the IVP in this case.

d) Finally, solve the IVP when $c = 51$ (here, we are in the situation of *over-damping*).

e) Use Maple or other software (Matlab, Wolfram alpha, etc) to create a display containing the graphs of all four solutions above, on the interval $0 \le t \le 4$. Print out a copy, and label which graph corresponds to which solution.

w6.2) (lab) (Where the magic algorithms of section 3.3 come from.) Let D represent differentiation with respect to x, that is

$$\mathrm{D} = \frac{d}{dx},\ \mathrm{D}^2 = \frac{d^2}{dx^2},$$

etc. Let a_0, a_1 be scalars. Use I for the identity operator, i.e. $\mathrm{I}(y) = y$ for all y. Then we can rewrite our familiar second order linear operator L as a linear combination of the operators I, D, D^2 (see text 3.3 for a discussion),

$$\begin{aligned} L\,y &= y'' + a_1 y' + a_0 y \\ &= D^2(y) + a_1 D(y) + a_0 I(y) \\ &= \left[D^2 + a_1 D + a_0 I\right](y). \end{aligned}$$

If the characteristic polynomial $p(r) = r^2 + a_1 r + a_0$ factors as

$$p(r) = (r - a)\cdot(r - b)$$

Then the operator L factors as a composition of first order operators (in either order):

$$D^2 + a_1 D + a_0 I = [D - a\,I] \circ [D - b\,I] = [D - b\,I] \circ [D - a\,I].$$

For distinct roots $a \neq b$ the basis functions for the homogeneous solution space to $L(y) = 0$ are $y_1 = e^{a x}$ and $y_2 = e^{b x}$ because

$$[D - a\,I]\,e^{ax} = 0 \quad \text{and} \quad [D - b\,I]\,e^{bx} = 0,$$

w6.2a) Verify that

$$[D - a\,I]\,e^{ax} = 0$$

w6.2b) Now suppose instead that the characteristic polynomial has a double root $a = b$, i.e. $p(r) = (r - a)^2$. Then

$$L = D^2 + a_1 D + a_0 I = [D - a\,I] \circ [D - a\,I].$$

Show that

$$[D - a\,I]\,x\,e^{a x} = e^{a x}.$$

w6.2c) Use parts (a),(b) and the composition of operators to deduce that

$$[D - a\,I] \circ [D - a\,I]\,x\,e^{a x} = 0.$$

(This explains why for double roots our homogeneous DE solution space basis is $\{e^{a x}, x\,e^{a x}\}$.

w6.2d) Show that for any differentiable function $f(x)$,

$$[D - a\,I]\,f(x)\,e^{a x} = f'(x)\,e^{a x}.$$

w6.2e) Explain using (d) why if the operator L of degree at least three has a factor

$$[D - a\,I] \circ [D - a\,I] \circ [D - a\,I] = [D - a\,I]^3$$

then the three functions $e^{a x}, x\,e^{a x}, x^2 e^{a x}$ all solve $L(y) = 0$.

w6.3) We explore why the general algorithm for finding a basis of solutions for constant coefficient homogeneous linear differential equations yields independent functions, for an illustrative example.

a) What 5^{th} order constant coefficient homogeneous linear DE has characteristic polynomial

$$p(r) = (r-3)^3\left(r^2 + 16\right) \ ?$$

b) According to our general algorithm, write down a basis for the 5-dimensional solution space to the DE in part a.

c) Prove that the five functions you wrote down in part b actually are linearly independent (so automatically span, so a basis). Hint: Use a combination of the techniques we discussed on Wednesday February 8, rather than attempting a Wronskian computation. Notice that the solutions exist on the entire real line $-\infty < x < \infty$ and some of them have different growth rates.