Math 2280-001
Week 5 concepts and homework, due Friday February 10

Recall that all problems are good for seeing if you can work with the underlying concepts; that the underlined problems are to be handed in; and that the Friday quiz will be drawn from all of these concepts and from these or related problems.

3.1
Solving initial value problems for linear homogeneous second order differential equations, given a basis for the solution space. Finding general solutions for constant coefficient homogeneous DE's by searching for exponential or other functions. Superposition for linear differential equations, and its failure for non-linear DE's.

1, **6**, (in 6 use initial values $y(0) = 10, y'(0) = -5$ rather than the ones in the text), **10**, 11, **12**, **14** (In 14 use the initial values $y(1) = 1, y'(1) = 7$ rather than the ones in the text.), 17, **18, 27**, 33, 39.

w5.3a) In 3.1.6 above, the text tells you that $y_1(x) = e^{2x}, y_2(x) = e^{-3x}$ are two independent solutions to the second order homogeneous differential equation $y'' + y' - 6y = 0$. Verify that you could have found these two exponential solutions via the following guessing algorithm: Try $y(x) = e^{rx}$ where the constant r is to be determined. Substitute this possible solution into the homogeneous differential equation and find the only two values of r for which $y(x)$ will satisfy the DE. (See Theorem 5 in the text.)

w5.3b) In 3.1.10 above, the text tells you that $y_1(x) = e^{5x}, y_2(x) = x\,e^{5x}$ are two independent solutions to the second order homogeneous differential equation $y'' - 10y' + 25y = 0$. Follow the procedure in part a of trying for solutions of the form $y(x) = e^{rx}$, and then use the repeated roots Theorem 6 in the text, to recover these two solutions.

3.2 Testing collections of functions for dependence and independence. Solving IVP's for homogeneous and non-homogeneous differential equations. Superposition.
1, 2, 5, **8**, 11,13, **16**, 21, **25**, 26

Here are two problems that explicitly connect ideas from sections 3.1-3.2 with linear algebra concepts from Math 2270

w5.4) Consider the 3^{rd} order homogeneous linear differential equation for $y(x)$

$$y'''(x) = 0$$

and let W be the solution space.
w5.4a) Use successive antidifferentiation to solve this differential equation. Interpret your results using vector space concepts to show that the functions $y_0(x) = 1, y_1(x) = x, y_2(x) = x^2$ are a basis for W. Thus the dimension of W is 3.
w5.4b) Show that the functions $z_0(x) = 1, z_1(x) = x - 2, z_2(x) = \frac{1}{2}(x-2)^2$ are also a basis for W.
Hint: If you verify that they solve the differential equation and that they're linearly independent, they will automatically span the 3-dimensional solution space and therefore be a basis.
w5.4c) Use a linear combination of the solution basis from part b, in order to solve the initial value

problem below. Notice how this basis is adapted to initial value problems at $x_0 = 2$, whereas for an IVP at $x_0 = 0$ the basis in a would have been easier to use.

$$y'''(x) = 0$$
$$y(2) = 7$$
$$y'(2) = -13$$
$$y''(2) = 5.$$

w5.5) Consider the three functions

$$y_1(x) = \cos(2x), \quad y_2(x) = \sin(2x), y_3(x) = \sin\left(2x - \frac{\pi}{6}\right).$$

w5.5a) Show that all three functions solve the differential equation

$$y'' + 4y = 0.$$

w.5b) The differential equation above is a second order linear homogeneous DE, so the solution space is 2-dimensional. Thus the three functions y_1, y_2, y_3 above must be linearly dependent. Find a linear dependency. (Hint: use a trigonometry addition angle formula.)

w5.5c) Explicitly verify that every initial value problem

$$y'' + 4y = 0$$
$$y(0) = b_1$$
$$y'(0) = b_2$$

has a solution of the form $y(x) = c_1\cos(2x) + c_2\sin(2x)$, and that c_1, c_2 are uniquely determined by b_1, b_2. (Thus $\cos(2x), \sin(2x)$ are a basis for the solution space of $y'' + 4y = 0$: every solution $y(x)$ has initial values that can be matched with a linear combination of y_1, y_2, but once the initial values match the solutions must agree by the uniqueness theorem, so y_1, y_2 span the solution space; y_1, y_2 are linearly independent because if $c_1\cos(2x) + c_2\sin(2x) = y(x) \equiv 0$ then $y(0) = y'(0) = 0$ so also $c_1 = c_2 = 0$.)

w5.5d) Find by inspection, particular solutions $y(x)$ to the two non-homogeneous differential equations

$$y'' + 4y = 28, \qquad y'' + 4y = -16x$$

Hint: one of them could be a constant, the other could be a multiple of x.

w5.5e) Use superposition (linearity) and your work from **c,d** to find the general solution to the non-homogeneous differential equation

$$y'' + 4y = 28 - 16x.$$

w5.5f) Solve the initial value problem, using your work above:

$$y'' + 4y = 28 - 16x$$
$$y(0) = 0$$
$$y'(0) = 0.$$

3.3: using the algorithm for finding the general solution to constant coefficient linear homogeneous differential equations: real roots, Euler's formula and complex roots, repeated roots; solving associated initial value problems.

3.3: 3, 9, 11, 23, 27.

NOTE: w5.6 and w5.7 are postponed until next week (Friday). We will talk about section 3.3 this Friday Feb. 10

w5.6) Do the following problems for homogeneous linear differential equations by hand. They are testing your ability to use the algorithm for finding bases for the solution spaces, based on the characteristic polynomial. Check your work with Maple (or other software). In Maple you will want to use the "dsolve" command to check differential equation solutions, and may want to use the "factor" command to check your factorizations of the characteristic polynomials. Hand in a printout of your computer verifications for the differential equation solutions, along with your written work.

a) Find the general solution to the differential equation for $y(x)$

$$y^{(3)} - 5\,y'' + 3\,y' + 9\,y = 0.$$

Hint: Find a root r_1 of the cubic characteristic polynomial, then divide it by $(r - r_1)$ to get a quotient quadratic polynomial.

b) Find the general solution to the differential equation for $x(t)$

$$x'' + 4\,x' + 13\,x = 0\,.$$

Hint: completing the square works well here - probably better than the quadratic formula.

c) Solve the initial value problem for the differential equation in b, with $x(0) = 0, x'(0) = 9$.

d) Find the general solution to the differential equation for $y(x)$

$$y^{(4)} - 8\,y' = 0.$$

e) Find the general solution to the differential equation for $y(x)$

$$y^{(5)} + 6\,y^{(3)} + 9\,y' = 0.$$

w5.7) Euler's formula

$$e^{i\,\theta} = \cos(\theta) + i\sin(\theta)$$

is extremely useful in higher mathematics/science/engineering. In class we discuss how this definition is motivated by Taylor series. Amazingly, the rule of exponents is true for such expressions. In other words

$$e^{i\,\alpha + i\,\beta} = e^{i\,\alpha}\,e^{i\,\beta}\,.$$

Check this identity by rewriting the left side as $e^{i\,(\alpha + \beta)}$ and then using Euler's formula to expand both sides. You'll notice that the identity is true because of the addition angle formulas for $\cos(\alpha + \beta)$ and $\sin(\alpha + \beta)$. (And so this gives a good way to recover the trig. identities if you happen to forget them.)

2.4-2.6 Numerical methods problems are at the end of this assignment.

w5.1) Runge-Kutta is based on Simpson's rule for numerical integration. Simpson's rule is based on the fact that for a subinterval $[d, d + h]$ of length h , the parabola $y = p(x)$ which passes through the points

(d, y_0), $\left(d + \frac{h}{2}, y_1\right)$, $(d + h, y_2)$ has integral

$$\int_d^{d+h} p(x)\, \mathrm{d}x = \frac{h}{6} \cdot (y_0 + 4\, y_1 + y_2)\,.$$

w5.1a) The integral approximation above follows from one on the interval $[-1, 1]$ by an affine change of variables. So first consider the interval $[-1, 1]$. We wish to find the parabolic function

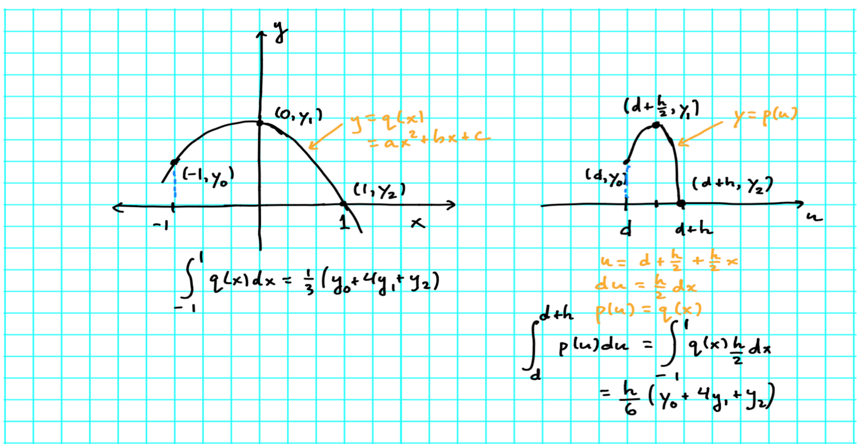
$$q(x) = a\,x^2 + b\,x + c$$

with unknown parameters a, b, c. We want $q(-1) = y_0,\ q(0) = y_1,\ q(1) = y_2$. This gives 3 equations in 3 unknowns, to find a, b, c in terms of y_0, y_1, y_2. Write down these linear equations and find a, b, c.

w5.1b) Compute $\int_{-1}^{1} q(x)\, \mathrm{d}x$ for thes values of a, b, c you find in part a, and verify the identity

$$\int_{-1}^{1} q(x)\, \mathrm{d}x = \frac{1}{3}\left(y_0 + 4y_1 + y_2\right)$$

Note, the formula for general interval follows from a change of variables, as indicated below:



Remark: If you've forgotten, or if you never talked about Simpson's rule in your Calculus class, here's how it goes: In order to approximate the definite integral of $f(x)$ on the interval $[a, b]$, you subdivide $[a, b]$ into n subintervals of width $\Delta x = \frac{b-a}{n} = h$. Then add the midpoints of each subinterval. Label these $x-$ values (incluidng midpoints) as

$$x_0 = a, x_1 = a + \frac{h}{2}, x_2 = a + h, x_3 = x_0 + \frac{3\,h}{2}, x_4 = x_0 + 2\,h, \ldots\, x_{2\,n} = x_0 + 2nh = b,$$

with corresponding $y-$values $y_i = f(x_i)$, $i = 0, \ldots 2\,n$. On each successive pair of intervals $\left[x_{2\,k}, x_{2\,k+1}\right]$ use the parabolic estimate

$$\int_{x_{2\,k}}^{x_{2\,k}+h} f(u)\, \mathrm{d}u \approx \frac{h}{6} \cdot \left(f(x_{2\,k}) + 4\,f(x_{2\,k+1}) + f(x_{2\,k+2})\right) = \frac{h}{6} \cdot \left(y_{2\,k} + 4\,y_{2\,k+1} + y_{2\,k+2}\right)$$

above, estimating the integral of f by the integral of the interpolating parabola on the subinterval. This yields the very accurate (for large enough n) Simpson's rule formula

$$\int_a^b f(x)\,dx \approx \frac{h}{6}\left(\left(y_0 + 4\,y_1 + y_2\right) + \left(y_2 + 4\,y_3 + y_4\right) + \ldots + \left(y_{2\,n-2} + 4\,y_{2\,n-1} + y_{2\,n}\right)\right),$$

i.e.

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6\,n}\left(y_0 + 4\,y_1 + 2\,y_2 + 4\,y_3 + 2\,y_4 + \ldots + 2\,y_{2\,n-2} + 4\,y_{2\,n-1} + y_{2\,n}\right).$$

http://en.wikipedia.org/wiki/Simpson's_rule

w5.2) (Famous numbers revisited, section 2.6, page 135, of text). The mathy numbers e, $\ln(2)$, π can be well-approximated using approximate solutions to differential equations. We illustrate this on Wednesday Feb. 4 for e, which is $y(1)$ for the solution to the IVP

$$y'(x) = y$$
$$y(0) = 1.$$

Apply Runge-Kutta with $n = 10, 20, 40$ ... subintervals, successively doubling the number of subintervals until you obtain the target number below - rounded to 9 decimal digits - twice in succession. We will do this in class for e, and you can modify that code if you wish.

w5.2a) $\ln(2)$ is $y(2)$, where $y(x)$ solves the IVP

$$y'(x) = \frac{1}{x}$$
$$y(1) = 0$$

(since $y(x) = \ln(x)$).

w5.2b) π is $y(1)$, where $y(x)$ solves the IVP

$$y'(x) = \frac{4}{x^2 + 1}$$
$$y(0) = 0$$

(since $y(x) = 4\arctan(x)$).

Note that in **a,b** you are actually "just" using Simpson's rule from Calculus, since the right sides of these DE's only depend on the variable x and not on the value of the function $y(x)$. For reference:

```
> Digits := 16 : #how many digits to use in floating point numbers and calcululations
  evalf(e); #evaluate the floating point of e
  evalf(π);
                         2.718281828459045
                         3.141592653589793                        (1)
```