Math 2280-001
Week 4 concepts and homework, due February 3.

Recall that problems which are not underlined are good for seeing if you can work with the underlying concepts; that the underlined problems are to be handed in; and that the Friday quiz will be drawn from all of these concepts and from these or related problems.

2.2: applications of population models.

week 4.1 (lab) Consider a bioreactor used by a yogurt factory to grow the bacteria needed to make yogurt. The growth of the bacteria is governed by the logistic equation

$$\frac{dP}{dt} = k \cdot P(M - P)$$

where P is the population in millions and t is the time in days. Recall that M is the carrying capacity of the reactor, and k is a constant that depends on the growth rate.
a) Through observation it is found that after a long time the population in the reactor stabilizes at 50 million bacteria, and that when the population of the reactor is 20 million bacteria the population increases at a rate of 12 million per day. From this, find k and M in the governing equation.
b) If the colony starts with a population of 10 million bacteria, how long will it take for the population to reach 80 % of carrying capacity?
c) Suppose the factory harvests the bacteria from the reactor once a week. The harvesting process takes a day, during which the reactor is not operational, leaving 6 days per week for the bacteria to grow in the reactor. The factory wants to maximize the amount of bacteria grown during these 6 days. To achieve this, $P'(t)$ should be at its maximum 3 days after harvesting. What initial population (after harvesting) gives the most growth over the 6-day period? What is the population change during this time?
d) Suppose the reactor is modified to allow for continual harvesting without shutting down the reactor. Let h be the rate at which the bacteria are harvested, in millions per day. Write down the new differential equation governing the bacteria population. What is the maximum rate of harvesting h that will not cause the population of bacteria to go extinct? (Harvesting at less than this rate will ensure that there is always a stable equilibrium point where P is positive.)

2.3: improved velocity-acceleration models:
constant, or constant plus linear drag forcing: **2, 3**, **9**, **10** (lab) **12** (lab)
quadratic drag: **13, 14,** 17

2.4-2.6: *numerical methods for approximating solutions to first order initial value problems.*
2.4: **4**: *Euler's method*
2.5: **4**: *improved Euler*
2.6: **4**: *Runge-Kutta*

The following problems are part of next week's homework, but we will be talking about these concepts on Wednesday (and possibly Friday) of this week:

week 5.1) Runge-Kutta is based on Simpson's rule for numerical integration. Simpson's rule is based on the fact that for a subinterval $[d, d+h]$ of length h, the parabola $y = p(x)$ which passes through the points (d, y_0), $\left(d + \frac{h}{2}, y_1\right)$, $(d+h, y_2)$ has integral

$$\int_d^{d+h} p(x)\, \mathrm{d}x = \frac{h}{6} \cdot (y_0 + 4\,y_1 + y_2).$$

w5.1a) The integral approximation above follows from one on the interval $[-1, 1]$ by an affine change of variables. So first consider the interval $[-1, 1]$. We wish to find the parabolic function

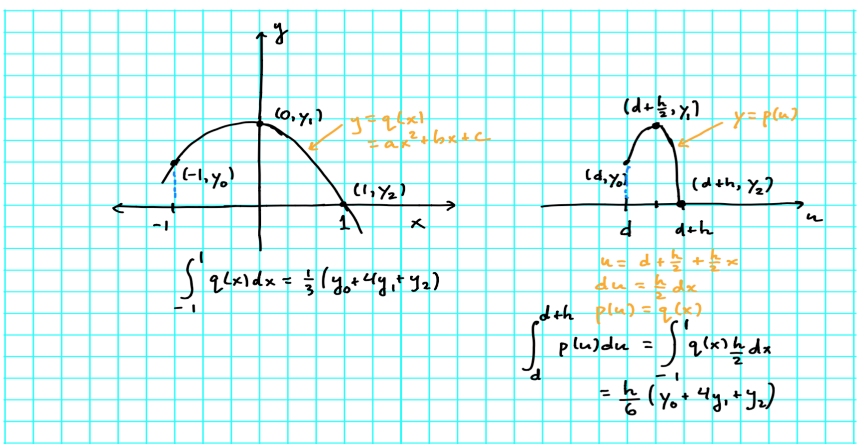
$$q(x) = a\,x^2 + b\,x + c$$

with unknown parameters a, b, c. We want $q(-1) = y_0$, $q(0) = y_1$, $q(1) = y_2$. This gives 3 equations in 3 unknowns, to find a, b, c in terms of y_0, y_1, y_2. Write down these linear equations and find a, b, c.

w5.1b) Compute $\int_{-1}^{1} q(x)\, \mathrm{d}x$ for thes values of a, b, c you find in part a, and verify the identity

$$\int_{-1}^{1} q(x)\, \mathrm{d}x = \frac{1}{3}\,(y_0 + 4y_1 + y_2)$$

Note, the formula for general interval follows from a change of variables, as indicated below:



Remark: If you've forgotten, or if you never talked about Simpson's rule in your Calculus class, here's how it goes: In order to approximate the definite integral of $f(x)$ on the interval $[a, b]$, you subdivide $[a, b]$ into n subintervals of width $\Delta x = \frac{b-a}{n} = h$. Then add the midpoints of each subinterval. Label these $x-$ values (incluidng midpoints) as

$$x_0 = a, x_1 = a + \frac{h}{2}, x_2 = a + h, x_3 = x_0 + \frac{3\,h}{2}, x_4 = x_0 + 2\,h, \ldots\, x_{2\,n} = x_0 + 2nh = b,$$

with corresponding $y-$values $y_i = f(x_i)$, $i = 0, \ldots 2\,n$. On each successive pair of intervals $\left[x_{2\,k}, x_{2\,k+1}\right]$

use the parabolic estimate

$$\int_{x_{2k}}^{x_{2k}+h} f(u)\,\mathrm{d}u \approx \frac{h}{6}\cdot\left(f(x_{2k})+4f(x_{2k+1})+f(x_{2k+2})\right)=\frac{h}{6}\cdot\left(y_{2k}+4y_{2k+1}+y_{2k+2}\right)$$

above, estimating the integral of *f* by the integral of the interpolating parabola on the subinterval. This yields the very accurate (for large enough *n*) Simpson's rule formula

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{h}{6}\left(\left(y_0+4y_1+y_2\right)+\left(y_2+4y_3+y_4\right)+\ldots+\left(y_{2n-2}+4y_{2n-1}+y_{2n}\right)\right),$$

i.e.

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{b-a}{6n}\left(y_0+4y_1+2y_2+4y_3+2y_4+\ldots+2y_{2n-2}+4y_{2n-1}+y_{2n}\right).$$

http://en.wikipedia.org/wiki/Simpson's_rule

w5.3) (Famous numbers revisited, section 2.6, page 135, of text). The mathy numbers e, $\ln(2)$, π can be well-approximated using approximate solutions to differential equations. We illustrate this on Wednesday Feb. 4 for e, which is $y(1)$ for the solution to the IVP

$$y'(x)=y$$
$$y(0)=1.$$

Apply Runge-Kutta with $n = 10, 20, 40$ ... subintervals, successively doubling the number of subintervals until you obtain the target number below - rounded to 9 decimal digits - twice in succession. We will do this in class for e, and you can modify that code if you wish.

a) $\ln(2)$ is $y(2)$, where $y(x)$ solves the IVP

$$y'(x)=\frac{1}{x}$$
$$y(1)=0$$

(since $y(x) = \ln(x)$).

b) π is $y(1)$, where $y(x)$ solves the IVP

$$y'(x)=\frac{4}{x^2+1}$$
$$y(0)=0$$

(since $y(x) = 4\arctan(x)$).

Note that in **a,b** you are actually "just" using Simpson's rule from Calculus, since the right sides of these DE's only depend on the variable x and not on the value of the function $y(x)$. For reference:

```
> Digits := 16 : #how many digits to use in floating point numbers and calcululations
  evalf(e); #evaluate the floating point of e
  evalf(π);
                                 2.718281828459045
                                 3.141592653589793                          (1)
>
```