Math 2280-001
Week 14 concepts and homework, due Wednesday April 26 at 6:00 p.m.

9.3 Fourier sine and cosine series
17, 19, 20

w14.1 Recall that an even function $f(t)$ is one for which $f(-t) = f(t)$ always hold, and that an odd function $g(t)$ is one for which $g(-t) = -g(t)$ always holds.
a) Let $f_1(t), f_2(t)$ be even. Prove $f_1(t) f_2(t)$ is also even.
b) Let $f(t)$ even, and $g(t)$ odd. Prove $f(t) g(t)$ is odd.
c) Let $g_1(t), g_2(t)$ be odd. Prove that $g_1(t) g_2(t)$ is even.

9.4: Understanding forced oscillation differential equations, with periodic forcing functions, via Fourier series expansions and superposition of particular solutions.
The following exercises are in the spirit of class notes/discussion on Wednesday April 12. Please make use of the particular solution table at the end of those notes.

w14.1 (lab) In class, when we were studying resonance (or not) with non-sinusoidal but periodic forcing functions, we discovered that resonance can happen whether or not the period of the forcing function is the natural period. Using the table of particular solutions included in this homework assignment, write down the general solution (particular plus homogeneous) to the following two forced oscillation problems. Check that in these examples, the forcing function with the natural period of 2π does not cause resonance, whereas the forcing function with period 6π does. ("Resonance" means that all solutions to the DE are the sum of a bounded function with one having linearly growing amplitude.)

a)

$$x''(t) + x(t) = \cos(t) + \sin\left(\frac{t}{3}\right).$$

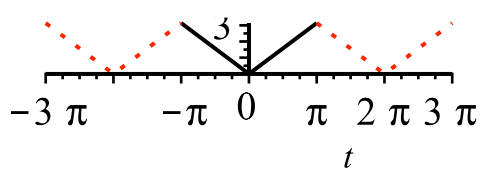
b)

$$x''(t) + x(t) = \cos(2t) - 3\sin(3t).$$

w14.2 (lab) Consider the forced oscillation problem

$$* \qquad x''(t) + x(t) = tent(t)$$

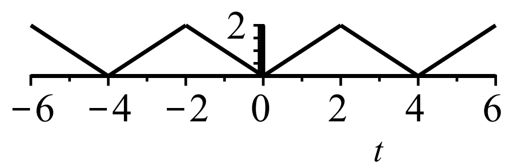
where $tent(t)$ is the 2π- periodic extension of the absolute value function, $|t|$ on the interval $[-\pi, \pi]$ to all of $\mathbb{R}$.



$$tent(t) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n\ odd} \frac{1}{n^2} \cos(n\,t)$$

We used convolution integrals on April 12 to show that solutions to this differential equation exhibit resonance. Use infinite superposition of particular solutions to find a particular solution to *, in order to explain precisely why resonance occurs in this problem: identify the term in the the Fourier series for $tent(t)$ that is causing the resonance, along with the part of solution that that corresponds to it.

w14.3 Consider the re-scaled tent function with period 4, which on the interval $[-2, 2]$ is given by $f_2(t) = |t|$.



a) Find the Fourier series for f_2, either from scratch or by rescaling the one for $tent(t)$.
b) Find a particular solution to

$$** \qquad x''(t) + x(t) = f_2(t)$$

via infinite superposition. (Resonance cannot occur in this case because none of the Fourier terms for the forcing function f_2 will be oscillating with natural angular frequency $\omega_0 = 1$.)

w14.4 (lab) Consider the slightly damped forced oscillation problem

$$*** \qquad x''(t) + 0.02\, x'(t) + 1.01\, x(t) = tent(t)$$

Because there is damping, pure resonance will not occur. Because the damping coefficient is near zero, however, the steady periodic solution will potentially have large oscillations - if one of the sinusoidal functions in the Fourier expansion of $tent(t)$ has angular frequency ω near ω_0 - a manifestation of practical resonance. (And from the Fourier series for $tent(t)$ you know that is the case here.)
Find $x_P(t) = x_{sp}(t)$, using infinite superposition, and identify the large-amplitude sinusoidal function in the sum which is responsible for the practical resonance.

w14.5 As we've been discussing, for the undamped forced oscillation problem

$$x''(t) + \omega_0^2\, x(t) = f(t),$$

where $f(t)$ has period $P = 2L$, then resonance will only occur if one of the sinusoidal functions in the Fourier series for f,

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n \frac{\pi}{L} t\right) + \sum_{n=1}^{\infty} b_n \sin\left(n \frac{\pi}{L} t\right)$$

has $\omega_n = \frac{n\pi}{L} = \omega_0$ (and with at least one of a_n, b_n non-zero). Solving for the period $P = 2L$ of the forcing function, this means

$$2L = P = n \frac{2\pi}{\omega_0}.$$

In other words, the forcing function period is exactly n times the natural period for homogeneous solutions. An example of this occured in Example 4 of the April 12 "resonance games", where the forcing function period was three times the natural period for homogeneous solutions. (We were pushing the swing every third time.) The forcing function had period 6π and was defined for $0 < t < 6\pi$ by

$$f_4(t) = \begin{cases} 1, & 0 < t < \pi \\ 0, & \pi < t < 6\pi \end{cases}$$

and the differential equation was

$$x''(t) + x(t) = f_4(t).$$

w14.5a) Find the Fourier series for f_4.

w14.5b) Check your Fourier coefficients using technology, as we've done several times in our class notes for different functions.

w14.5c) Use infinite superposition to verify that resonance occurs in this problem, and identify which term in the the forcing function and corresponding particular solution piece is responsible for it.

6.3 Ecological models: predators and competition

1,2, 4, 5, **6, 7**, 11, 12, **13**.

w14.6 Use pplane or other technology to create a phase diagram for the nonlinear system in problems 4-7 above and verify that your linearization work in 6, 7 is consistent with this phase portrait.

6.4 Nonlinear mechanical systems

9, 10, **11, 12**, 13, **14**, 15

Particular solutions from Chapter 3 or Laplace transform table:

$$x''(t) + \omega_0^2 x(t) = A\,sin(\omega t)$$

$$x_P(t) = \frac{A}{\omega_0^2 - \omega^2} \sin(\omega t) \quad \text{when } \omega \neq \omega_0$$

$$x_P(t) = -\frac{t}{2\,\omega_0} A \cos(\omega_0 t) \quad \text{when } \omega = \omega_0$$

....

$$x''(t) + \omega_0^2 x(t) = A \cos(\omega t)$$

$$x_P(t) = \frac{A}{\omega_0^2 - \omega^2} \cos(\omega t) \quad \text{when } \omega \neq \omega_0$$

$$x_P(t) = \frac{t}{2\,\omega_0} A \sin(\omega_0 t) \quad \text{when } \omega = \omega_0$$

....

$$x'' + c\,x' + \omega_0^2 x = A \cos(\omega t) \quad c > 0$$

$$x_P(t) = x_{sp}(t) = C \cos(\omega t - \alpha)$$

with

$$C = \frac{A}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + c^2\omega^2}} \; .$$

$$\cos(\alpha) = \frac{\omega_0^2 - \omega^2}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + c^2\omega^2}}$$

$$\sin(\alpha) = \frac{c\,\omega}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + c^2\omega^2}} \; .$$

....

$$x'' + c\,x' + \omega_0^2 x = A \sin(\omega t) \quad c > 0$$

$$x_P(t) = x_{sp}(t) = C \sin(\omega t - \alpha)$$

with

$$C = \frac{A}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + c^2\omega^2}} \; .$$

$$\cos(\alpha) = \frac{\omega^2 - \omega_0^2}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + c^2\omega^2}}$$

$$\sin(\alpha) = \frac{c\,\omega}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + c^2\omega^2}}$$