Math 2280-001
Week 12 concepts and homework
sections 5.7, 7.1-7.3
Due Tuesday April 11 (at the start of lab)

5.7 *Nonhomogeneous Linear Systems: undetermined coefficients and variation of parameters.*
1, 13, 15, 19, 23.

In some sense the examples/methods in the problem w12.1 contain pretty much all of the course so far - since any linear differential equation or system of differential equations is equivalent to a linear first order system. In practice the methods we've learned earlier are less computationally intensive, but it's nice to know there's a larger framework that contains everything...

w12.1 (lab) Consider the input-output IVP from the second midterm

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 20 \\ 40 \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

w12.1a) Using matrix exponentials, the solution to

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}} + \underline{\boldsymbol{f}}(t)$$
$$\underline{\boldsymbol{x}}(0) = \underline{\boldsymbol{x}}_0$$

is given by

$$\underline{\boldsymbol{x}}(t) = e^{tA}\left(\underline{\boldsymbol{x}}_0 + \int_0^t e^{-sA}\underline{\boldsymbol{f}}(s)\,ds\right).$$

Verify that this recovers the correct solution

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix} - 10\,e^{-2t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

w12.1b) Alternately, when A is diagonalizable, as it is in this case, we can change output variables to reduce to a simpler system: Let

$$A\,S = S\,\Lambda$$

where the columns of S are an eigenbasis for $\mathbb{R}^n$ or $\mathbb{C}^n$ and Λ is the diagonal matrix of corresponding eigenvalues. Then

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}} + \underline{\boldsymbol{f}}(t)$$
$$\underline{\boldsymbol{x}}(0) = \underline{\boldsymbol{x}}_0$$

is equivalent to

$$S^{-1}\underline{\boldsymbol{x}}' = S^{-1}A\,\underline{\boldsymbol{x}} + S^{-1}\underline{\boldsymbol{f}}(t).$$

For

$$\underline{\boldsymbol{u}}(t) := S^{-1}\underline{\boldsymbol{x}}(t)$$
$$\underline{\boldsymbol{u}}'(t) = S^{-1}\underline{\boldsymbol{x}}'(t)$$

this yields the diagonal system

$$\underline{\boldsymbol{u}}'(t) = S^{-1}AS\,\underline{\boldsymbol{u}}(t) + S^{-1}\underline{\boldsymbol{f}}(t)$$

i.e. the IVP

$$\underline{\boldsymbol{u}}'(t) = \Lambda\,\underline{\boldsymbol{u}}(t) + S^{-1}\underline{\boldsymbol{f}}(t)$$
$$\underline{\boldsymbol{u}}(0) = S^{-1}\underline{\boldsymbol{x}}_0$$

After finding $\underline{\boldsymbol{u}}(t)$, the original solution $\underline{\boldsymbol{x}}(t) = S\,\underline{\boldsymbol{u}}(t)$. Rework the tank IVP using this method.

<u>**w12.2**</u> (lab). A clean proof that when matrices A, B commute, then $e^{A+B} = e^A e^B$:
Use the product rule to show that if A, B commute, then

$$X(t) = e^{tA} e^{tB}$$

solves

$$X'(t) = (A+B)X$$
$$X(0) = I$$

Deduce that

$$e^{t(A+B)} = e^{tA} e^{tB}$$

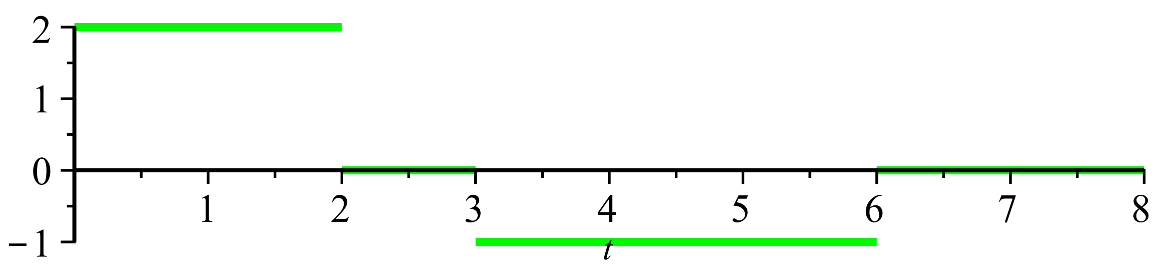
for all t, in particular for $t = 1$. Hint: Use the fact A and B commute not only with each other, but therefore also with their matrix exponentials.

7.1: Laplace transforms and inverse transforms.
Use the definition of Laplace transform and integration techniques to compute Laplace transforms of $f(t)$*:*
1, 3, 7, 9
<u>**w12.3)**</u> Use the definition of Laplace transform

$$\mathcal{L}\{f(t)\}(s) = \int_0^{\infty} f(t)e^{-st}\,dt$$

and integration by parts to compute the Laplace transform $F(s)$ of $f(t) = 4\,t\,e^{2t}$.

<u>**w12.4)**</u> Consider the function $f(t)$ which is zero for $t > 6$, is piecewise constant, and has this graph



Break the Laplace transform integral from zero to infinity into the sum of integrals over four subintervals (two of the integrals will be zero and use antidifferentiation for the other two), in order to compute the Laplace transform $F(s)$ of $f(t)$.

Use Laplace transform table, linearity (and useful trig identities) to compute Laplace transforms and inverse Laplace transforms.
13, 17, 19, 23, 29

w12.5a) Use the Laplace transform table in the front cover of the text, algebra, and the linearity of Laplace transform, to compute the Laplace transform $F(s)$ for

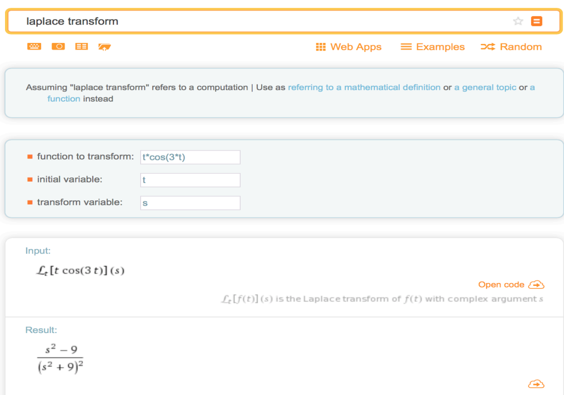
$$f(t) = 2\,e^{-t}\sin(3\,t) - 2\,e^{-3\,t}(2 - 3\,t) - 4\,t^2.$$

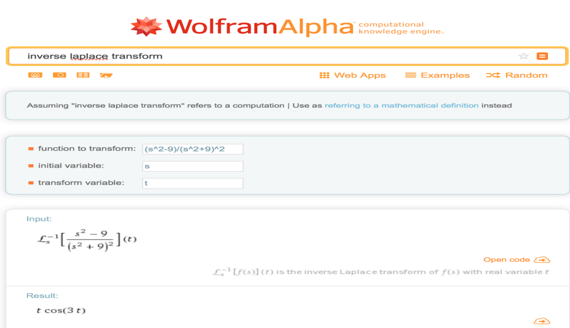
b) Check your answer with technology. (Wolfram alpha does everything.)

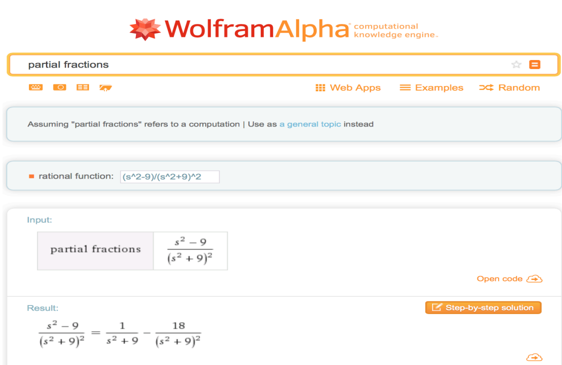
w12.6a) Use the Laplace transform table and linearity to compute the inverse Laplace transform $g(t)$ of

$$G(s) = \frac{3}{s+4} + \frac{1}{(s-4)^2} - \frac{2\,s+6}{s^2+4} + \frac{2\,s+3}{(s^2+9)^2}\,.$$

b) Check your answer with technology. (Wolfram alpha does everything but often uses complex exponentials so you've got to be good with Euler's formula to check your work.)







7.2: Transforming and solving initial value problems via Laplace transforms;
3, 7, 9,19.

w12.7) Use Laplace transforms to solve the underdamped initial value problem

$$x''(t) + 4\,x'(t) + 13\,x(t) = 0$$
$$x(0) = 1$$
$$x'(0) = 4$$

w12.8a) Use Laplace transforms (and partial fractions) to solve the initial value problem for $x(t)$:

$$x''(t) + 6\,x'(t) + 9\,x(t) = 30\cos(3\,t)$$
$$x(0) = 0$$
$$x'(0) = 0$$

b) Check your answer with technology.

w12.9). Use Laplace transforms to solve the same input-output IVP as in **w12.1**
Hint: If you take the Laplace transform of the two differential equations you will get a system of two algebraic equations for

$$\underline{\mathbf{X}}(s) = \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix}$$

which you can solve. After some partial fractions decomposition of $X_1(s), X_2(s)$ you will be able to find the inverse Laplace transforms $x_1(t), x_2(t)$. It may be helpful to use the matrix vector form of the differential equation to find $\underline{\mathbf{X}}(s)$:

$$\underline{\mathbf{x}}'(t) = A\,\underline{\mathbf{x}}(t) + \underline{\mathbf{f}}(t)$$

then it follows after transforming each equation that

$$s\,\underline{\mathbf{X}}(s) - \underline{\mathbf{x}}_0 = A\,\underline{\mathbf{X}}(s) + \underline{\mathbf{F}}(s)$$

i.e.

$$(s\,I - A)\underline{\mathbf{X}}(s) = \underline{\mathbf{x}}_0 + \underline{\mathbf{F}}(s)$$
$$\underline{\mathbf{X}}(s) = (s\,I - A)^{-1}\left(\underline{\mathbf{x}}_0 + \underline{\mathbf{F}}(s)\right)$$

7.2-7.3: Laplace transform table entries; partial fractions to simplify F(s); the translation theorem with completing the square, to identify inverse Laplace transforms; applying these and other techniques to initial value problems.

7.2: **20**

7.3: 3, 7, 9, 17, **20**, 27, **30**, 32, 34.

w12.10) With access to a Laplace transform table it is possible to very quickly recover the general solutions to key mechanical oscillation problems (some of which are real messy to derive with Chapter 3 techniques). Do this for

a) undamped forced oscillation, $\omega \neq \omega_0$:

$$x''(t) + \omega_0^2\, x(t) = \frac{F_0}{m} \cos(\omega\, t)$$
$$x(0) = x_0$$
$$x'(0) = v_0$$

b) undamped forced oscillation, $\omega = \omega_0$:

$$x''(t) + \omega_0^2\, x(t) = \frac{F_0}{m} \cos(\omega_0\, t)$$
$$x(0) = x_0$$
$$x'(0) = v_0$$