2b) Use these linearity properties to show that

Theorem 0 the solution space to the homogeneous second order linear DE

$$L(y) = y'' + p(x)y' + q(x)y = 0$$

is closed under addition and scalar multiplication, i.e. it is a subspace. Notice that this is the "same" proof one uses to show that the solution space to a homogeneous matrix equation $A\,\underline{\mathbf{x}} = \underline{\mathbf{0}}$ is a subspace.

Let $L(y_1) = 0$, $L(y_2) = 0$.
(i.e. Let y_1, y_2 be solthns to the homogeneous DE.)

then $L(y_1 + y_2) = L(y_1) + L(y_2) = 0 + 0 = 0$
so $y_1 + y_2$ is a homog. solthn.

also $L(cy_1) = cL(y_1) = c \cdot 0 = 0$
so cy_1 is a homog. solthn.

in 2270
"homogeneous solthn space"
i.e. $\{\vec{x} \in \mathbb{R}^n \text{ s.t. } A\vec{x} = \vec{0}\}$
is a subspace:
Let $\vec{z}, \vec{w}$ be homogeneous solthns, i.e. $A\vec{z} = \vec{0}$, $A\vec{w} = \vec{0}$
then $A(\vec{z} + \vec{w}) = A\vec{z} + A\vec{w} = \vec{0} + \vec{0} = \vec{0}$
$A(c\vec{z}) = cA\vec{z} = c\vec{0} = \vec{0}$

Exercise 3) As an example, find the solution space to the following homogeneous differential equation for $y(x)$

$$y'' + 2y' = 0$$

on the x-interval $-\infty < x < \infty$. Notice that the solution space is the span of two functions. Hint: This is really a first order DE for $v = y'$.

$$v' + 2v = 0 \quad *$$

$$e^{2x}(v' + 2v) = e^{2x} \cdot 0 = 0$$

$$\frac{d}{dx}(e^{2x}v) = 0$$

$$e^{2x}v = C$$

$$v = Ce^{-2x}$$

$$y'(x) = Ce^{-2x}$$

$$\Longrightarrow y(x) = -\frac{C}{2}e^{-2x} + D$$

$$y(x) = c_1 e^{-2x} + c_2 \cdot 1$$

solthn space is $\operatorname{span}\{e^{-2x}, 1\}$

two ways that one sees subspace:

implicit way $\longrightarrow$ (1) solthn space to $L(y) = 0$, where L is linear.
of y's

explicit way $\longrightarrow$ (2) $\operatorname{span}\{y_1, y_2, \dots y_n\} = \{c_1y_1 + c_2y_2 + \dots + c_ny_n \mid c_1, c_2, \dots c_n \in \mathbb{R}\}$

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(1) solthn space of $\vec{x}$'s s.t. $A\vec{x} = \vec{0}$
"nullspace"
(2) $\operatorname{span}\{\vec{v}_1, \dots \vec{v}_n\}$

Exercise 4) Use the linearity properties to show

Theorem 1 All solutions to the nonhomogeneous second order linear DE

$$L(y) = y'' + p(x)y' + q(x)y = f(x)$$

are of the form $y = y_P + y_H$ where y_P is any single particular solution and y_H is some solution to the homogeneous DE. (y_H is called y_c, for complementary solution, in the text). Thus, if you can find a single particular solution to the nonhomogeneous DE, and all solutions to the homogeneous DE, you've actually found all solutions to the nonhomogeneous DE.

① if y_P is a particular soltn
i.e. $L(y_P) = f$
let $L(y_H) = 0$.
then $L(y_P + y_H) = L(y_P) + L(y_H) = f + 0 = f$
so $y_P + y_H$ solves the nonhomog DE

② if y_Q is any (other) soltn to $L(y) = f$
write $y_Q = y_P + (y_Q - y_P)$
$\Rightarrow L(y_Q) = L(\text{—})$
$f = L(y_P) + L(y_Q - y_P)$
$\cancel{f} = \cancel{f} + L(y_Q - y_P)$
$0 = L(y_Q - y_P)$, so $y_Q = y_P + y_H$, with $y_H = y_Q - y_P$.

2270 for $A\vec{x} = \vec{b}$
if $A\vec{x}_p = \vec{b}$
then every soltn to $A\vec{x} = \vec{b}$ is of the form $\vec{x} = \vec{x}_p + \vec{x}_H$
where $\vec{x}_H$ is in the nullspace of A

Theorem 2 (Existence-Uniqueness Theorem): Let $p(x), q(x), f(x)$ be specified continuous functions on the interval I, and let $x_0 \in I$. Then there is a unique solution $y(x)$ to the initial value problem

$$y'' + p(x)y' + q(x)y = f(x) \quad *$$
$$y(x_0) = b_0$$
$$y'(x_0) = b_1$$

and $y(x)$ exists and is twice continuously differentiable on the entire interval I.

for now, we'll just believe this →

Chptr 4
why this true (later)
2nd order DE is actually equivalent to a 1st order system:
If y solves $*$
then
$$\begin{bmatrix} y \\ y' \end{bmatrix}' = \begin{bmatrix} y' \\ y'' \end{bmatrix}$$
$$\begin{bmatrix} y \\ y' \end{bmatrix}' = \begin{bmatrix} y' \\ f - py' - qy \end{bmatrix}$$
of form $(x_1 = y, x_2 = y')$

IVP
$$\begin{cases} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} x_2 \\ f - px_2 - qx_1 \end{bmatrix} \\ \begin{bmatrix} x_1(x_0) \\ x_2(x_0) \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} \end{cases}$$

$- \; y_p + y_H \qquad - \;$ existence, uniqueness

Exercise 5) Verify Theorems 1 and 2 for the interval $I = (-\infty, \infty)$ and the IVP $\quad y(x)$

$$y'' + 2y' = 3$$
$$y(0) = b_0$$
$$y'(0) = b_1$$

Let $v = y'$

$$v' + 2v = \boxed{3}$$
$$e^{2x}(v' + 2v) = 3e^{2x}$$
$$\frac{d}{dx}\, e^{2x} v = 3e^{2x}$$
$$e^{2x} v = \frac{3}{2} e^{2x} + C$$
$$v = \frac{3}{2} + Ce^{-2x}$$
$$y' = \frac{3}{2} + Ce^{-2x}$$
$$y = \frac{3}{2}x + \frac{C}{-2} e^{-2x} + D$$
$$y(x) = \underbrace{\boxed{\frac{3}{2}x}}_{y_p} + \underbrace{c_1 e^{-2x} + c_2}_{y_H}$$

$\to$ general soltn to $y'' + 2y' = 0$ (Friday)

$(\, y_p'' + 2y_p' = 0 + 2 \cdot \frac{3}{2} = 3 \; \checkmark \,)$

note, $\frac{3}{2}x + 2e^{-2x} + 7$ is another particular soltn, so I could also have written

$$y(x) = \frac{3}{2}x + 2e^{-2x} + 7 + d_1 e^{-2x} + d_2$$

Thm 2 $\quad y(x) = \frac{3}{2}x + c_1 e^{-2x} + c_2$

$$\Rightarrow y'(x) = \frac{3}{2} - 2c_1 e^{-2x}$$

for IVP: $\quad y(0) = b_0 = 0 + c_1 + c_2$

$$y'(0) = b_1 = \frac{3}{2} - 2c_1$$

for each initial value vector $\begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$ there is a unique $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

Unlike in the previous example, and unlike what was true for the first order linear differential equation

$$y' + p(x)y = q(x)$$

there is not a clever integrating factor formula that will always work to find the general solution of the second order linear differential equation

$$y'' + p(x)y' + q(x)y = f(x).$$

Rather, we will usually resort to vector space theory and algorithms based on clever guessing to solve these differential equations. It will help to know

Theorem 3: The solution space to the second order homogeneous linear differential equation

$$y'' + p(x)y' + q(x)y = 0$$

is 2-dimensional.

This Theorem is illustrated in Exercise 2 that we completed earlier. Theorem 3 and the techniques we'll actually be using going forward are illustrated by

Exercise 6) Consider the homogeneous linear DE for $y(x)$

$$y'' - 2y' - 3y = 0$$

6a) Find two exponential functions $y_1(x) = e^{rx}$, $y_2(x) = e^{\rho x}$ that solve this DE. Deduce that arbitrary linear combinations of y_1, y_2 also solve the DE.

6b) Show that every IVP

$$\left.\begin{aligned} y'' - 2y' - 3y &= 0 \\ y(0) &= b_0 \\ y'(0) &= b_1 \end{aligned}\right\}$$

$y(x) = c_1 e^{3x} + c_2 e^{-x}$ want

$\Rightarrow y(0) = c_1 + c_2 = b_0$

$y'(0) = 3c_1 - c_2 = b_1$

$$\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$$

can be solved with a unique linear combination $y(x) = c_1 y_1(x) + c_2 y_2(x)$.

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{-4}\begin{bmatrix} -1 & -1 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$$

6c) Use your work from part b to explain why the solution space is two-dimensional.

6d) Now consider the nonhomogeneous DE

§3.4

$$y'' - 2y' - 3y = 9$$

Notice that $\boxed{y_P(x) = -3}$ is a particular solution. Use this information and superposition (linearity) to find the solution to the initial value problem

$$\begin{aligned} y'' - 2y' - 3y &= 9 \\ y(0) &= 6 \\ y'(0) &= -2. \end{aligned}$$

$$\Rightarrow \boxed{\begin{aligned} y &= y_P + y_H \\ y &= -3 + c_1 e^{3x} + c_2 e^{-x} \end{aligned}}$$

$y' - ky = 0$

$y' = ky$

$y = Ce^{kx}$

6a) $y'' - 2y' - 3y = 0.$ *

try $y = e^{rx}$ (find "r").

$y' = re^{rx}$

$y'' = r^2 e^{rx}$

so $y'' - 2y' - 3y = r^2 e^{rx} - 2re^{rx} - 3e^{rx}$

$= e^{rx}[r^2 - 2r - 3]$

$= 0$ (zero fcn)

iff r is a root of $p(r) = r^2 - 2r - 3$

$= (r-3)(r+1)$

$r = 3, -1.$

$\left[\begin{aligned} y_1 &= e^{3x} \\ y_2 &= e^{-x} \end{aligned}\right.$ solve *

(you could check).

$\Rightarrow y_H(x) = c_1 e^{3x} + c_2 e^{-x}$ solve *.

c) Why soln is is 2-dim'l.

claim: $y_1(x) = e^{3x}$, $y_2(x) = e^{-x}$ is a basis for soln space

- span
- linearly independent.

We showed in (b) that every IVP $\begin{cases} y'' - 2y' - 3y = 0 & \text{DE} \\ y(0) = b_0 \\ y'(0) = b_1 \end{cases}$

has a soln $y(x) = c_1 y_1 + c_2 y_2$.

Let $z(x)$ be any soln to DE.

call $z(0) = b_0$, $z'(0) = b_1$. pick $y(x) = c_1 y_1 + c_2 y_2$ so that $y(0) = b_0$, $y'(0) = b_1$ } c_1, c_2 uniquely ~~be~~ determined by b_0, b_1

uniqueness theorem

$\Longrightarrow$ on $y(x) = z(x)$.

$\Rightarrow$ soln space $= \text{span}\{y_1, y_2\}$.

if $c_1 y_1 + c_2 y_2 \equiv 0$

y

then $y(x)$ is a homog. soln.

$y(0) = 0$, $y'(0) = 0$ $\Longrightarrow c_1 = c_2 = 0$.

d) $\begin{cases} y'' - 2y' - 3y = 9 \\ y(0) = 6 \\ y'(0) = -2. \end{cases}$

$y(x) = -3 + c_1 e^{3x} + c_2 e^{-x}$

$y'(x) = 3c_1 e^{3x} - c_2 e^{-x}$.

I.C's: $y(0) = 6 = -3 + c_1 + c_2$

$y'(0) = -2 = 3c_1 - c_2$

$$\begin{bmatrix} 9 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\frac{1}{-4} \begin{bmatrix} -1 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 9 \\ -2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$-\frac{1}{4} \begin{bmatrix} -7 \\ -29 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$