Math 2280-001
Fri Feb 3

• I'll post my version of filled in Wed. notes
• think about HW timing.

• Most likely, we did not finish Wednesday's notes. This will also be an opportunity to summarize the three numerical methods for solving DE's that we discussed: i.e. Euler, Improved Euler, and Runge Kutta, and for you to ask any questions you might have related to the 2.4-2.6 material. There is an application problem in this upcoming week's homework that will use numerical methods.

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• Then, begin Chapter 3, which is about higher order linear differential equations and applications.

3.1 Second order linear differential equations, and vector space theory connections to Math 2270:

Definition: A vector space is a collection of objects together with an "addition" operation "+", and a "scalar multiplication" operation, so that the rules below all hold.

(α) Whenever $f, g \in V$ then $f + g \in V$. (closure with respect to addition)
(β) Whenever $f \in V$ and $c \in \mathbb{R}$, then $c \cdot f \in V$. (closure with respect to scalar multiplication)

As well as:
(a) $f + g = g + f$ (commutative property) : at any x, $(f+g)(x) = f(x)+g(x) = g(x)+f(x) = (g+f)(x)$
(b) $f + (g + h) = (f + g) + h$ (associative property) ✓
(c) $\exists\ \underline{0} \in V$ so that $f + \underline{0} = f$ is always true. "0" : $0(x) = 0$
(d) $\forall f \in V\ \exists -f \in V$ so that $f + (-f) = 0$ (additive inverses) : $(-f)(x) = -(f(x))$
(e) $c \cdot (f + g) = c \cdot f + c \cdot g$ (scalar multiplication distributes over vector addition)
(f) $(c_1 + c_2) \cdot f = c_1 \cdot f + c_2 \cdot f$ (scalar addition distributes over scalar multiplication)
(g) $c_1 \cdot (c_2 \cdot f) = (c_1 c_2) \cdot f$ (associative property)
(h) $1 \cdot f = f$, $(-1) \cdot f = -f$, $0 \cdot f = 0$ (these last two actually follow from the others).

Examples you've seen in Math 2270:
(1) $\mathbb{R}^m$, with the usual vector addition and scalar multiplication, defined component-wise $\vec{x} \in \mathbb{R}^n$
(2) subspaces W of $\mathbb{R}^m$, which satisfy (α),(β), and therefore automatically satisfy (a)-(h), because the vectors in W also lie in $\mathbb{R}^m$.

or fixed subinterval of $\mathbb{R}$

Maybe you've also seen ...
Exercise 1) In Chapter 3 we focus on the vector space

$$V = C(\mathbb{R}) := \{ f : \mathbb{R} \to \mathbb{R} \ s.t.\ f \text{ is a continuous function} \}.$$

and its subspaces. Verify that the vector space axioms for linear combinations are satisfied for this space of functions. Recall that the function $f + g$ is defined by $\boxed{(f + g)(x) := f(x) + g(x)}$ and the scalar multiple $c f(x)$ is defined by $\boxed{(c f)(x)} := \boxed{c f(x)}$. What is the zero vector for functions?

e.g in Calculus: $(f+g)' = f' + g'$
$(cf)' = cf'$

relation to $\vec{z} \in \mathbb{R}^n$
$$\vec{z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$
scalar n components, one for each i, $1 \le i \le n$.

for f a function, get a "component" for each x

what is a linear combination of $f_1, f_2, \ldots f_n$ is any sum of scalar multiples
i.e. any $c_1 f_1 + c_2 f_2 + \cdots + c_n f_n$
where $c_1, c_2 \ldots c_n \in \mathbb{R}$

Recall that the vector space axioms are exactly the arithmetic rules we use to work with linear combination equations. In particular the following concepts are defined in any vector space V.

- the span of a finite collection of functions $\{f_1, f_2, \ldots f_n\}$. $= \text{span}\{f_1, f_2, \ldots f_n\}$ = set of all linear combos of $f_1 \ldots f_n$ $= \{c_1 f_1 + c_2 f_2 + \cdots + c_n f_n$ s.t. each $c_j \in \mathbb{R}\}$
- linear independence/dependence for a collection of functions $f_1, f_2, \ldots f_n$.
- subspaces of V : subset of V that is closed under + & scalar. (subspace is a vector space)
- bases and dimension for finite dimensional subspaces. (The function space V itself is infinite dimensional, meaning that no finite collection of functions spans it.)

$\{f_1, f_2, \ldots f_n\}$ is linearly dependent: at least one f_j is a linear combo of the others
i.e. $c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0$ for some choice $c_1, c_2 \ldots c_n$ where not all $c_j = 0$

independent (not dependent)
i.e. $c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0$
$\iff c_1 = c_2 = \cdots = c_n = 0$.

a basis for W is a collection of vectors $\{f_1, f_2 \ldots f_n\}$ that span W and are linearly independent

# of vectors in any basis of W.

Definition: A second order linear differential equation for a function $y(x)$ is a differential equation that can be written in the form

$$A(x)y'' + B(x)y' + C(x)y = F(x) .$$

We search for solution functions $y(x)$ defined on some specified interval I of the form $a < x < b$, or (a, ∞), $(-\infty, a)$ or (usually) the entire real line $(-\infty, \infty)$. In this chapter we assume the function $A(x) \neq 0$ on I, and divide by it in order to rewrite the differential equation in the standard form

$$y'' + p(x)y' + q(x)y = f(x) .$$

contrast with 1st order linear
$y' + p(x)y = q(x)$

Definition: The DE above is called homogeneous if the right hand side $f(x)$ is the zero function, $f(x) \equiv 0$. If f is not the zero function, the DE is called nonhomogeneous (or inhomogeneous).

One reason the DE above is called linear is that the "operator" L defined by

$$L(y) := y'' + p(x)y' + q(x)y$$

satisfies the so-called linearity properties

- (1) $L(y_1 + y_2) = L(y_1) + L(y_2)$
- (2) $L(c\,y) = c\,L(y),\ c \in \mathbb{R}$.

2270 $L(\vec{x}) = A\vec{x}$
$A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$
$A(c\vec{x}) = c\,A\vec{x}$
homog solutions: $A\vec{x} = \vec{0}$.

(Recall that the matrix multiplication function $L(\underline{x}) := A\,\underline{x}$ satisfies the analogous properties. Any time we have have a transformation L satisfying (1),(2), we say it is a linear transformation.)

Exercise 2a) Check the linearity properties (1),(2) for the differential operator $L(y) := y'' + p(x)y' + q(x)y$.

$L(y_1 + y_2) = (y_1 + y_2)'' + p(x)(y_1 + y_2)' + q(x)(y_1 + y_2)$
$= y_1'' + y_2'' + p(x)(y_1' + y_2') + q(x)(y_1 + y_2)$

$L(y_1)$ $L(y_2)$

$= L(y_1) + L(y_2)$ ✓

$L(cy) = (cy)'' + p(x)(cy)' + q(x)(cy)$
$= cy'' + c\,p(x)y' + c\,q(x)y$
$= c(L(y))$

ex $L(y) = y'' + 3y' + 2y$
$L(x^2) = 2 + 3\cdot 2x + 2x^2$
$= 2 + 6x + 2x^2$
the fcn $y(x) = x^2$

$L(e^{-x}) = e^{-x} + 3(-e^{-x}) + 2e^{-x}$
$= 0$
$L(e^{-x}) = 0$ means e^{-x} solves homogeneous DE
$y'' + 3y' + 2y = 0$

2b) Use these linearity properties to show that

Theorem 0 the solution space to the homogeneous second order linear DE

$$L(y) = y'' + p(x)y' + q(x)y = 0$$

is closed under addition and scalar multiplication, i.e. it is a subspace. Notice that this is the "same" proof one uses to show that the solution space to a homogeneous matrix equation $A\,\underline{x} = \underline{0}$ is a subspace.

Let $L(y_1) = 0$, $L(y_2) = 0$.
(i.e. Let y_1, y_2 be solns to the homogeneous DE.)

then $L(y_1 + y_2) = L(y_1) + L(y_2) = 0 + 0 = 0$
so $y_1 + y_2$ is a homog. soltn.

also $L(cy_1) = c\,L(y_1) = c \cdot 0 = 0$
so cy_1 is a homog. soltn.

in 2270
"homogeneous soltn space"
i.e. $\{\vec{x} \in \mathbb{R}^n \text{ s.t. } A\vec{x} = \vec{0}\}$
is a subspace:
Let $\vec{z}, \vec{w}$ be homogeneous solns, i.e. $A\vec{z} = \vec{0}$, $A\vec{w} = \vec{0}$
then $A(\vec{z} + \vec{w}) = A\vec{z} + A\vec{w} = \vec{0} + \vec{0} = \vec{0}$
$A(c\vec{z}) = c\,A\vec{z} = c\vec{0} = \vec{0}$

Exercise 3) As an example, find the solution space to the following homogeneous differential equation for $y(x)$

$$y'' + 2\,y' = 0$$

on the x-interval $-\infty < x < \infty$. Notice that the solution space is the span of two functions. Hint: This is really a first order DE for $v = y'$.

$$v' + 2v = 0 \quad *$$

$$e^{2x}(v' + 2v) = e^{2x} \cdot 0 = 0$$

$$\frac{d}{dx}\left(e^{2x} v\right) = 0$$

$$e^{2x} v = C$$

$$v = Ce^{-2x}$$

$$y'(x) = Ce^{-2x}$$

$$\Rightarrow y(x) = -\frac{C}{2} e^{-2x} + D$$

$$y(x) = c_1 e^{-2x} + c_2 \cdot 1$$

soltn space is span $\{e^{-2x}, 1\}$

Exercise 4) Use the linearity properties to show

Theorem 1 All solutions to the nonhomogeneous second order linear DE

$$y'' + p(x)y' + q(x)y = f(x)$$

are of the form $y = y_P + y_H$ where y_P is any single particular solution and y_H is some solution to the homogeneous DE. (y_H is called y_c, for complementary solution, in the text). Thus, if you can find a single particular solution to the nonhomogeneous DE, and all solutions to the homogeneous DE, you've actually found all solutions to the nonhomogeneous DE.

Theorem 2 (Existence-Uniqueness Theorem): Let $p(x), q(x), f(x)$ be specified continuous functions on the interval I, and let $x_0 \in I$. Then there is a unique solution $y(x)$ to the initial value problem

$$y'' + p(x)y' + q(x)y = f(x)$$
$$y(x_0) = b_0$$
$$y'(x_0) = b_1$$

and $y(x)$ exists and is twice continuously differentiable on the entire interval I.

Exercise 5) Verify Theorems 1 and 2 for the interval $I = (-\infty, \infty)$ and the IVP

$$
\begin{aligned}
y'' + 2y' &= 3 \\
y(0) &= b_0 \\
y'(0) &= b_1
\end{aligned}
$$

Unlike in the previous example, and unlike what was true for the first order linear differential equation

$$y' + p(x)y = q(x)$$

there is <u>not</u> a clever integrating factor formula that will always work to find the general solution of the second order linear differential equation

$$y'' + p(x)y' + q(x)y = f(x).$$

Rather, we will usually resort to vector space theory and algorithms based on clever guessing to solve these differential equations. It will help to know

<u>**Theorem 3**:</u> The solution space to the second order homogeneous linear differential equation

$$y'' + p(x)y' + q(x)y = 0$$

is 2-dimensional.

This Theorem is illustrated in <u>Exercise 2</u> that we completed earlier. Theorem 3 <u>and</u> the techniques we'll actually be using going forward are illustrated by

<u>Exercise 6)</u> Consider the homogeneous linear DE for $y(x)$

$$y'' - 2y' - 3y = 0$$

<u>6a)</u> Find two exponential functions $y_1(x) = e^{rx}$, $y_2(x) = e^{\rho x}$ that solve this DE. Deduce that arbitrary linear combinations of y_1, y_2 also solve the DE.

<u>6b)</u> Show that every IVP

$$\begin{aligned} y'' - 2y' - 3y &= 0 \\ y(0) &= b_0 \\ y'(0) &= b_1 \end{aligned}$$

can be solved with a unique linear combination $y(x) = c_1 y_1(x) + c_2 y_2(x)$.

<u>6c)</u> Use your work from part <u>b</u> to explain why the solution space is two-dimensional.

<u>6d</u>) Now consider the nonhomogeneous DE

$$y'' - 2y' - 3y = 9$$

Notice that $y_P(x) = -3$ is a particular solution. Use this information and superposition (linearity) to find the solution to the initial value problem

$$\begin{aligned} y'' - 2y' - 3y &= 9 \\ y(0) &= 6 \\ y'(0) &= -2. \end{aligned}$$