$\Rightarrow y - (d_1 y_1 + d_2 y_2 + \cdots + d_k y_k)$ is in nullspace
so it $= c_1 z_1 + c_2 z_2 + \cdots + c_m z_m$ some c_j's
$\Rightarrow y = c_1 z_1 + \cdots + c_m z_m + d_1 y_1 + d_2 y_2 + \cdots + d_k y_k$ ∎

Math 2280-001
Fri Feb 24

Finish Wednesday notes §3.5
Fri: finish undetermined coef's
(rank + nullity theorem).
variation of parameters

to do these notes Monday

Section 3.6: forced oscillations in mechanical systems (and as we shall see in section 3.7, also in electrical circuits) overview:

We study solutions $x(t)$ to

$$m\,x'' + c\,x' + k\,x = \boxed{F_0 \cos(\omega\,t)}$$

forcing fcn.

using section 3.5 undetermined coefficients algorithms.

- undamped $(c = 0)$:

  In this case the complementary homogeneous differential equation for $x(t)$ is

$$m\,x'' + k\,x = 0$$
$$x'' + \frac{k}{m}x = 0$$
$$x'' + \omega_0^2\,x = 0$$

  which has simple harmonic motion solutions

$$x_H(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) = C_0 \cos(\omega_0 t - \alpha)\,.$$

So for the non-homongeneous DE the section 5.5 method of undetermined coefficients implies we can find particular and general solutions as follows:

$L(x) = F_0 \cos\omega t$ $\quad$ $L: \text{span}\{\cos\omega t\}$ $\to \text{span}\{\cos\omega t\}$

- $\omega \ne \omega_0 := \sqrt{\frac{k}{m}}$ $\Rightarrow x_P = A\cos(\omega\,t)$ because only even derivatives, we don't need

sin $(\omega\,t)$ terms !!

$$\Rightarrow x = x_P + x_H = A\cos(\omega\,t) + C_0 \cos(\omega_0 t - \alpha_0)\,.$$

- $\omega \ne \omega_0$ but $\omega \approx \omega_0$, $A \approx C_0$ Beating!
- $\omega = \omega_0$ case 2 section 3.5 undetermined coefficients; since

$$p(r) = r^2 + \omega_0^2 = (r + i\omega_0)^1 (r - i\omega_0)^1$$

our undetermined coefficients guess is

$$x_P = t^1 \left(A\cos(\omega_0\,t) + B\sin(\omega_0\,t)\right)$$
$$\Rightarrow x = x_P + x_H = C\,t\cos(\omega\,t - \alpha) + C_0\cos(\omega_0 t - \alpha_0)\,.$$

("pure" resonance!)

- damped $(c > 0)$: in all cases $x_P = A\cos(\omega\,t) + B\sin(\omega\,t) = C\cos(\omega\,t - \alpha)$ (because the roots of the characteristic polynomial are never purely imagninary $\pm i\,\omega$ when $c > 0$).
  - underdamped: $x = x_P + x_H = C\cos(\omega\,t - \alpha) + e^{-p\,t} C_1 \cos(\omega_1 t - \alpha_1)\,.$
  - critically-damped: $x = x_P + x_H = C\cos(\omega\,t - \alpha) + e^{-p\,t}(c_1 t + c_2)\,.$
  - over-damped: $x = x_P + x_H = C\cos(\omega\,t - \alpha) + c_1 e^{-r_1 t} + c_2 e^{-r_2 t}\,.$

- in all three cases on the previous page, $x_H(t) \rightarrow 0$ exponentially and is called the <u>transient solution</u> $x_{tr}(t)$ (because it disappears as $t \rightarrow \infty$).

$x_P(t)$ as above is called the <u>steady periodic solution</u> $x_{sp}(t)$ (because it is what persists as $t \rightarrow \infty$, and because it's periodic).

- if c is small enough and $\omega \approx \omega_0$ then the amplitude C of $x_{sp}(t)$ can be large relative to $\frac{F_0}{m}$, and the system can exhibit <u>practical resonance.</u> This can be an important phenomenon in electrical circuits, where amplifying signals is important. We don't generally want pure resonance or practical resonance in mechanical configurations.

<u>Forced undamped oscillations</u>:

<u>Exercise 1a)</u> Solve the initial value problem for $x(t)$:

$$\left.\begin{aligned} x'' + 9x &= 80\cos(5t) \\ x(0) &= 0 \\ x'(0) &= 0\,. \end{aligned}\right\}.$$

1b) This superposition of two sinusoidal functions <u>is</u> periodic because there is a common multiple of their (shortest) periods. What is this (common) period?

1c) Compare your solution and reasoning with the display at the bottom of this page.

$x_H(t) = A\cos 3t + B\sin 3t$

$x'' + 9x = 0$

$p(r) = r^2 + 9 = 0$

$r = \pm 3i$

$x_p(t) = d\cos 5t$

$x_p'' + 9x_p = -25d\cos 5t + 9d\cos 5t$

$= -16d\cos 5t = 80\cos 5t \Rightarrow d = -5.$

$x(t) = -5\cos 5t + A\cos 3t + B\sin 3t$

$x(0) = 0 = -5 + A \Rightarrow A = 5$

$x'(0) = 3B = 0 \Rightarrow B = 0.$

$\boxed{x(t) = -5\cos 5t + 5\cos 3t}$

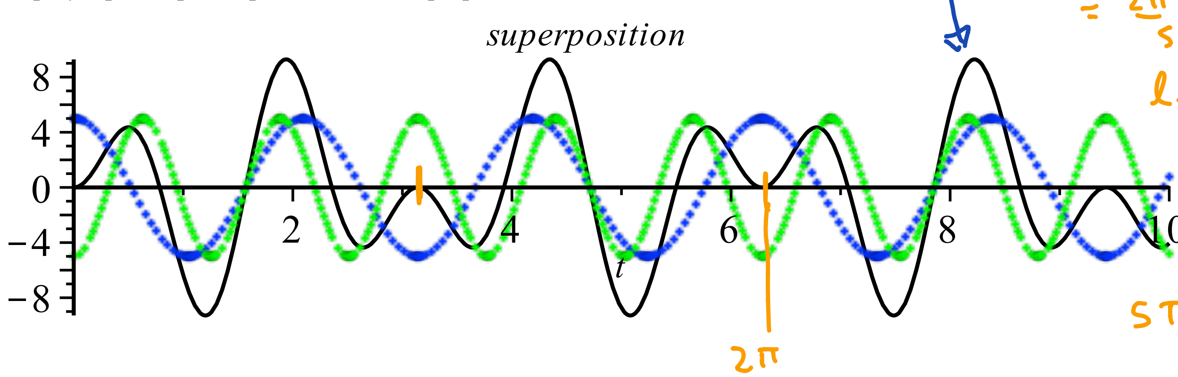
$T_1 =$ period. $= \frac{2\pi}{5}$ $\quad T_2 = \frac{2\pi}{3}$

l.c.m. $\left(\frac{2\pi}{5}, \frac{2\pi}{3}\right)$

$\frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5},$

$5T_1 = \underbrace{2\pi} = 3T_2$

```
> with(plots) :
> plot1 := plot(-5·cos(5·t), t = 0..10, color = green, style = point) :
  plot2 := plot(5·cos(3·t), t = 0..10, color = blue, style = point) :
  plot3 := plot(-5·cos(5·t) + 5·cos(3·t), t = 0..10, color = black) :
  display({plot1, plot2, plot3}, title='superposition');
```



In general:

undamped forced IVP, $\omega \neq \omega_0$, with letters

$$\begin{cases} x'' + \frac{k}{m} x = \frac{F_0}{m}\cos\omega t \\ x(0) = x_0 \\ x'(0) = v_0 \end{cases}$$

$$\begin{aligned} +\tfrac{k}{m}\,(\ x_p &= A\cos\omega t\ ) \\ +0\,(\ x_p' &= -A\omega\sin\omega t\ ) \\ +1\,(\ x_p'' &= -A\omega^2\cos\omega t\ ) \end{aligned}$$

$$L(x_p) = \cos\omega t\, A\left[\underset{\uparrow\ \omega_0^2}{\tfrac{k}{m}} - \omega^2\right]$$

deduce $A(\omega_0^2 - \omega^2) = \frac{F_0}{m}$

$A = \frac{F_0}{m}/(\omega_0^2 - \omega^2)$

So,

$$\boxed{x_p(t) = -\frac{F_0}{m}\frac{1}{(\omega^2-\omega_0^2)}\cos\omega t\ .}$$

Note $x_H(t) = A\cos\omega_0 t + B\sin\omega_0 t$.

so, by plugging in or observation IVP solution is

$$\boxed{x(t) = \frac{+F_0}{m(\omega^2-\omega_0^2)}\left(\cos\omega_0 t - \cos\omega t\right) + x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t}$$

check-NR!

$x = x_p + x_H$

$x(t) = -\frac{F_0}{m}\frac{1}{(\omega^2-\omega_0^2)}\cos\omega t + c_1\cos\omega_0 t + c_2\sin\omega_0 t$

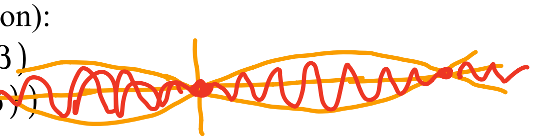
$x(0) = x_0 = -\frac{F_0}{m}\frac{1}{\omega^2-\omega_0^2} + c_1$

$\Rightarrow c_1 = x_0 + \frac{F_0}{m}\frac{1}{\omega^2-\omega_0^2}$

$x'(0) = v_0 = c_2\omega_0 \Rightarrow c_2 = \frac{v_0}{\omega_0}$

There is an interesting <u>beating</u> phenomenon for $\omega \approx \omega_0$ (but still with $\omega \neq \omega_0$). This is explained analytically via trig identities, and is familiar to musicians in the context of superposed sound waves (which satisfy the homogeneous linear "wave equation" partial differential equation):

$$\begin{aligned} \cos(\alpha-\beta) - \cos(\alpha+\beta) &= \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) \\ &\quad - (\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)) \\ &= 2\sin(\alpha)\sin(\beta)\ . \end{aligned}$$



$\alpha - \beta = \omega_0 t$ $\qquad$ $\alpha + \beta = \omega t$

Set $\alpha = \frac{1}{2}(\omega + \omega_0)t,\ \beta = \frac{1}{2}(\omega - \omega_0)t$ in the identity above, to rewrite the first term in $x(t)$ as a product rather than a difference:

$\approx \omega_0$ $\qquad$ $\omega \approx \omega_0$ long period $\left|\frac{2\pi}{\frac{1}{2}(\omega-\omega_0)}\right| = \left|\frac{4\pi}{\omega-\omega_0}\right|$

$$x(t) = \frac{F_0}{m\left(\omega^2 - \omega_0^2\right)}\,2\sin\left(\frac{1}{2}(\omega+\omega_0)t\right)\sin\left(\frac{1}{2}(\omega-\omega_0)t\right) + x_0\cos(\omega_0 t) + \frac{v_0}{\omega_0}\sin(\omega_0 t)\ .$$

In this product of sinusoidal functions, the first one has angular frequency and period close to the original angular frequencies and periods of the original sum. But the second sinusoidal function has small angular frequency and long period, given by

$\omega \to \omega_0$ $\qquad$ $\sin\theta \approx \theta$

$$\text{angular frequency: } \frac{1}{2}(\omega - \omega_0), \qquad \text{period: } \frac{4\pi}{|\omega - \omega_0|}\ .$$

$$\frac{F_0}{m\,\cancel{(\omega-\omega_0)}(\omega+\omega_0)}\,\cancel{2}\,\sin\omega_0 t\left(\underbrace{\cancel{\tfrac{1}{2}}\cancel{(\omega-\omega_0)}t \pm \tfrac{1}{2^3}\tfrac{1}{3!}\left(\cancel{(\omega-\omega_0)}t\right)^3}\right) + x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t$$

$$\text{as } \omega \to \omega_0,\ x(t) \to \underbrace{\frac{F_0}{2m\omega_0}(\sin\omega_0 t)(t)}_{x_p(t)} + x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t$$

We will call half that period the beating period, as explained by the next exercise:

$$\text{beating period: } \frac{2\pi}{|\omega-\omega_0|}, \qquad \text{beating amplitude: } \frac{2F_0}{m|\omega^2-\omega_0^2|}.$$

Exercise 2a) Use one of the formulas on the previous page to write down the IVP solution $x(t)$ to

$$x'' + 9x = 80\cos(3.1\,t)$$
$$x(0) = 0$$
$$x'(0) = 0.$$

$\omega_0 = 3$ $\quad \omega = 3.1$ $\quad F_0 = 80$ $\quad m = 1$

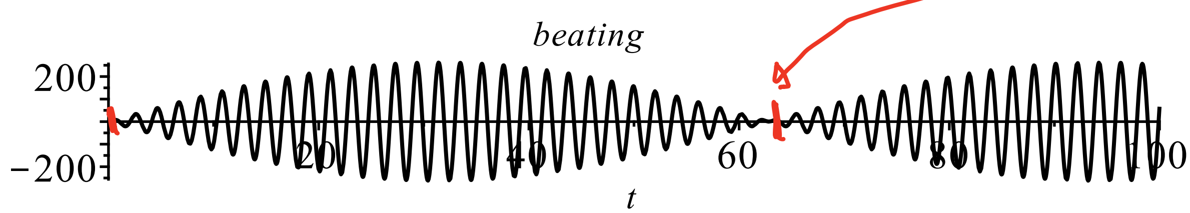
2b) Compute the beating period and amplitude. Compare to the graph shown below.

$$x(t) = \frac{80}{1\cdot((3.1)^2-3^2)}\, 2\sin(3.05t)(\sin(.05t)) + 0 + 0$$
$$= \frac{160}{.61}\sin(3.05t)\underbrace{\sin(.05t)}$$

$T = \frac{2\pi}{.05} = 40\pi$

beating period $= 20\pi$

> $plot(262.3\cdot\sin(3.05\cdot t)\sin(.05\cdot t),\ t=0..100,\ color=black,\ title=\text{`beating`});$



Resonance:

Resonance! $\omega = \omega_0$ (and the limit as $\omega \to \omega_0$)

$$\begin{cases} x'' + \omega_0^2 x = \frac{F_0}{m}\cos\omega_0 t \\ x(0) = x_0 \\ x'(0) = v_0 \end{cases}$$

using § 5.5, guess

$+\omega_0^2(\quad x_p = t\,(A\cos\omega_0 t + B\sin\omega_0 t)\quad)$

$0\,(\quad x_p' = t\,(-A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t) + A\cos\omega_0 t + B\sin\omega_0 t\quad)$

$+1\,(\quad x_p'' = t\,(-A\omega_0^2\cos\omega_0 t - B\omega_0^2\sin\omega_0 t) + [-A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t]\,2\quad)$

$$L(x_p) = t\,(0) + 2\,[-A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t] \overset{\text{want}}{=} \frac{F_0}{m}\cos\omega_0 t$$

Deduce $A = 0$, $B = \frac{F_0}{2m\omega_0}$

$$x_p(t) = \underbrace{\frac{F_0}{2m\omega_0}\, t\sin\omega_0 t}_{\text{sats } x(0)=0,\ x'(0)=0}$$

, so IVP soln is

$$\boxed{x(t) = \frac{F_0}{2m\omega_0}\, t\sin\omega_0 t + x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t}$$

(You can also get this solution by letting $\omega \to \omega_0$ in the beating formula.)

Exercise 3a) Solve the IVP

$$x'' + 9x = 80\cos(3t)$$
$$x(0) = 0$$
$$x'(0) = 0\,.$$

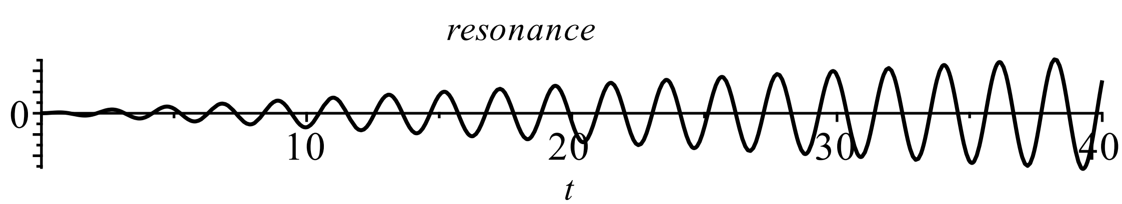
$\omega = \omega_0 = 3$

Either use the general solution formula above this exercise and substitute in the appropriate values for the various terms, or use this as a chance to practice variation of parameters for the particular solution.

$x(t) = \frac{80}{6}\, t \sin 3t$

3b) Compare the solution graph below with the beating graph in exercise 2.

> $plot\left(\frac{40}{3} t\cdot\sin(3\cdot t),\ t = 0..40,\ color = black,\ title = `resonance`\right);$



>

Damped forced oscillations $(c > 0)$ for $x(t)$:

$$m x'' + c x' + k x = F_0 \cos(\omega t)$$

Undetermined coefficients for $x_P(t)$:

$$\begin{aligned} k & \left[x_P = A\cos(\omega t) + B\sin(\omega t)\right] \\ + c & \left[x_P' = -A\omega\sin(\omega t) + B\omega\cos(\omega t)\right] \\ + m & \left[x_P'' = -A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t)\right] . \end{aligned}$$

$$\begin{aligned} L(x_P) &= \cos(\omega t)\left(kA + cB\omega - mA\omega^2\right) \\ &\quad + \sin(\omega t)\left(kB - cA\omega - mB\omega^2\right) . \end{aligned} \qquad \overset{\text{want}}{=} \cos\omega t\,(F_0) + \sin\omega t\,(0)$$

Collecting and equating coefficients yields the matrix system

$$\begin{bmatrix} k - m\omega^2 & c\omega \\ -c\omega & k - m\omega^2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} F_0 \\ 0 \end{bmatrix},$$

which has solution

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{\left(k - m\omega^2\right)^2 + c^2\omega^2} \begin{bmatrix} k - m\omega^2 & -c\omega \\ c\omega & k - m\omega^2 \end{bmatrix} \begin{bmatrix} F_0 \\ 0 \end{bmatrix} = \frac{F_0}{\left(k - m\omega^2\right)^2 + c^2\omega^2} \begin{bmatrix} k - m\omega^2 \\ c\omega \end{bmatrix}$$

In amplitude-phase form this reads

$$x_P = A\cos(\omega t) + B\sin(\omega t) = C\cos(\omega t - \alpha)$$

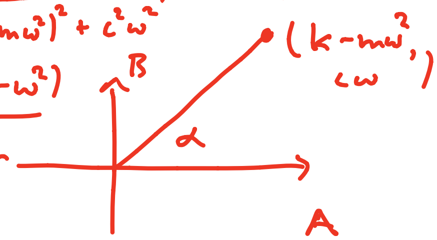
with

$$C = \frac{F_0}{\sqrt{\left(k - m\omega^2\right)^2 + c^2\omega^2}} . \quad \text{(Check!)} \quad = \frac{F_0}{(k-m\omega^2)^2 + c^2\omega^2}\sqrt{(k-m\omega^2)^2 + c^2\omega^2}$$

$$\cos(\alpha) = \frac{k - m\omega^2}{\sqrt{\left(k - m\omega^2\right)^2 + c^2\omega^2}} = \frac{m(\omega_0^2 - \omega^2)}{\text{denom}}$$

$$\sin(\alpha) = \frac{c\omega}{\sqrt{\left(k - m\omega^2\right)^2 + c^2\omega^2}} .$$

$k - m\omega^2$
$= m\left(\frac{k}{m} - \omega^2\right)$
$= m\left(\omega_0^2 - \omega^2\right)$



And the general solution $x(t) = x_P(t) + x_H(t)$ is given by

- underdamped: $x = x_{sp} + x_{tr} = C\cos(\omega t - \alpha) + e^{-pt}C_1\cos\left(\omega_1 t - \alpha_1\right)$.
- critically-damped: $x = x_{sp} + x_{tr} = C\cos(\omega t - \alpha) + e^{-pt}\left(c_1 t + c_2\right)$.
- over-damped: $x = x_{sp} + x_{tr} = C\cos(\omega t - \alpha) + c_1 e^{-r_1 t} + c_2 e^{-r_2 t}$.

Important to note:

- The amplitude C in x_{sp} can be quite large relative to $\dfrac{F_0}{m}$ if $\omega \approx \omega_0$ and $c \approx 0$, because the denominator will then be close to zero. This phenomenon is practical resonance.
- The phase angle α is always in the first or second quadrant.

Exercise 4) (a cool M.I.T. video.) Here is practical resonance in a mechanical mass-spring demo. Notice that our math on the previous page exactly predicts when the steady periodic solution is in-phase and when it is out of phase with the driving force, for small damping coefficient c! Namely, for c small, when $\boxed{\omega^2 << \omega_0^2}$ we have α near zero (in phase) for x_{sp}, because $\sin(\alpha) \approx 0$, $\cos(\alpha) \approx 1$; when $\omega^2 >> \omega_0^2$ we have α near π (out of phase), because $\sin(\alpha) \approx 0$, $\cos(\alpha) \approx -1$; for $\omega \approx \omega_0$, α is near $\frac{\pi}{2}$, because $\sin(\alpha) \approx 1$, $\cos(\alpha) \approx 0$.

http://www.youtube.com/watch?v=aZNnwQ8HJHU

Exercise 5) Solve the IVP for $x(t)$:

$$x'' + 2x' + 26x = 82\cos(4t)$$
$$x(0) = 6$$
$$x'(0) = 0.$$

Solution:

$$x(t) = \sqrt{41}\cos(4t - \alpha) + \sqrt{10}\, e^{-t}\cos(5t - \beta)$$
$$\alpha = \arctan(0.8),\ \beta = \arctan(-3).$$

```
> with(DEtools) :
> dsolve( {x''(t) + 2·x'(t) + 26·x(t) = 82·cos(4·t), x(0) = 6, x'(0) = 0});
```

$$x(t) = -3\,e^{-t}\sin(5t) + e^{-t}\cos(5t) + 5\cos(4t) + 4\sin(4t) \qquad \textbf{(4)}$$