A vector space theorem like the one for the base case, except for $L : V \rightarrow W$, combined with our understanding of how to factor constant coefficient differential operators (as in last week's homework) leads to an extension of the method of undetermined coefficients, for right hand sides which can be written as sums of functions having the indicated forms below. See the discussion in section 3.5 of the text, pages 190-191 of the new edition of our text, and the table 3.5.1, reproduced here.

Method of undetermined coefficients (extended case): If L has a factor $(D - r\,I)^s$ and $e^{r\,x}$ is also associated with (a portion of) the right hand side $f(x)$ then the corresponding guesses you would have made in the "base case" need to be multiplied by x^s , as in Exercise 7. (If you understood the homework problem last week about factoring L into composition of terms like $(D - r\,I)^s$, then you have an inkling of why this recipe works. If you didn't understand that last week problem, there's another one this week so you get a second chance. :-)) You may also need to use superposition, as in Exercise 4, if different portions of $f(x)$ are associated with different exponential functions.

Extended case of undetermined coefficients

$f(x)$	y_P	$s > 0$ when $p(r)$ has these roots:
$P_m(x) = b_0 + b_1 + \ldots + b_m x^m$	$x^s\left(c_0 + c_1 x + c_2 x^2 + \ldots + c_m x^m\right)$	$r = 0$
$b_1 \cos(\omega\,x) + b_2 \sin(\omega\,x)$	$x^s\left(c_1 \cos(\omega\,x) + c_2 \sin(\omega\,x)\right)$	$r = \pm\, i\,\omega$
$e^{a\,x}\left(b_1 \cos(\omega\,x) + b_2 \sin(\omega\,x)\right)$	$x^s e^{a\,x}\left(c_1 \cos(\omega\,x) + c_2 \sin(\omega\,x)\right)$	$r = a \pm i\omega$
$b_0 e^{a\,x}$	$x^s c_0 e^{a\,x}$	$r = a$
$\left(b_0 + b_1 + \ldots + b_m x^m\right)e^{a\,x}$	$x^s\left(c_0 + c_1 x + c_2 x^2 + \ldots + c_m x^m\right)e^{a\,x}$	$r = a$

Exercise 8) Set up the undetermined coefficients particular solutions for the examples below. When necessary use the extended case to modify the undetermined coefficients form for y_P. Use technology to check if your "guess" form was right.

$$L(y) := y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = f$$

8a) $y''' + 2y'' = \boxed{x^2 + 6x}$

y_H contains $e^{0x} = 1$, $x e^{0x} = x$

(So the characteristic polynomial for $L(y) = 0$ is $r^3 + 2r^2 = r^2(r+2) = (r-0)^2(r+2)$.)

> dsolve(y'''(x) + 2·y''(x) = x^2 + 6·x, y(x));

$$y(x) = \underbrace{\frac{1}{24}x^4 + \frac{5}{12}x^3 + \frac{1}{4}e^{-2x}\_C1 + \frac{5}{8}x^2}_{y_P} + \underbrace{\_C2\, x + \_C3}_{y_H} \quad \textbf{(2)}$$

>

$$y_P = x^2\left(d_1 x^2 + d_2 x + d_3\right) = d_1 x^4 + d_2 x^3 + d_3 x^2$$

$L = D^3 + 2D^2$

$[D+2I] \circ [D] \circ [D]$

$L : V \to W = \mathrm{span}\{1, x, x^2\}$

$V = \mathrm{span}\{x^2, x^3, x^4\}$, $f \in W$

8b) $y'' - 4y' + 13y = 4e^{2x}\sin(3x)$?

(So the characteristic polynomial for $L(y) = 0$ is

$r^2 - 4r + 13 = (r-2)^2 + 9 = (r - 2 + 3i)(r - 2 - 3i)$.) $r = 2 \pm 3i$

> dsolve(y''(x) − 4·y'(x) + 13·y(x) = 4·e^{2·x}·sin(3·x), y(x));

$$y(x) = \underbrace{e^{2x}\sin(3x)\_C2 + e^{2x}\cos(3x)\_C1}_{y_H} \underbrace{- \frac{2}{3}e^{2x}\cos(3x)\, x}_{y_P} \quad \textbf{(3)}$$

>

$y_H = c_1 e^{2x}\sin 3x + c_2 e^{2x}\cos 3x$

(for $y'' - 4y' + 13y = 0$).

$$y_P = x^1\left(d_1 e^{2x}\sin 3x + d_2 e^{2x}\cos 3x\right)$$

$$\mathrm{span}\{x e^{2x}\sin 3x,\ x e^{2x}\cos 3x\} = V \xrightarrow{L} W = \mathrm{span}\{e^{2x}\sin 3x,\ e^{2x}\cos 3x\} \ni f$$

$$L = D^2 - 4D + 13I = [D - (2-3i)I] \circ [D - (2+3i)I]$$

$$L\left(x e^{(2+3i)x}\right)$$

$$= [D - (2-3i)I]\, e^{(2+3i)x}$$

$$= (2+3i)\,e^{(2+3i)x} - (2-3i)\,e^{(2+3i)x}$$

So, $$L\left(xe^{(2+3i)x}\right) = 6i\,e^{(2+3i)x}$$

expand each side into real & imaginary fcns, and use linearity of L:

$$L(xe^{2x}\cos 3x) + i\,L(xe^{2x}\sin 3x)$$

$$= 6i\,(e^{2x}\cos 3x + i\,e^{2x}\sin 3x)$$

equating real fcns:

$$L(xe^{2x}\cos 3x) = -6\,e^{2x}\sin 3x$$

$$L(xe^{2x}\sin 3x) = 6\,e^{2x}\cos 3x.$$

So for $L(y_p) = 4e^{2x}\sin 3x$

we may choose $\boxed{y_p = -\frac{2}{3}\,x\,e^{2x}\cos 3x}$

Not every right hand side is amenable to finding particular solutions via undetermined coefficients. Luckily there is a more general (but technically messier) way that will always work:

Variation of Parameters: The advantage of this method is that is always provides a particular solution, even for non-homogeneous problems in which the right-hand side doesn't fit into a nice finite dimensional subspace preserved by L, and even if the linear operator L is not constant-coefficient. The formula for the particular solutions can be somewhat messy to work with, however, once you start computing.

Here's the formula: Let $y_1(x), y_2(x), \ldots y_n(x)$ be a basis of solutions to the homogeneous DE

$$L(y) := y^{(n)} + p_{n-1}(x)y^{(n-1)} + \ldots + p_1(x)y' + p_0(x)y = 0\,.$$

Then $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) + \ldots + u_n(x)y_n(x)$ is a particular solution to

$$L(y) = f$$

provided the coefficient functions (aka "varying parameters") $u_1(x), u_2(x), \ldots u_n(x)$ have derivatives satisfying the Wronskian matrix equation

$$\begin{bmatrix} y_1 & y_2 & \ldots & y_n \\ y_1' & y_2' & \ldots & y_n' \\ \ldots & \ldots & \ldots & \ldots \\ y_1^{(n-1)} & y_2^{(n-1)} & \ldots & y_n^{(n-1)} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ : \\ u_n' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ : \\ f \end{bmatrix}$$

n=2 case

Here's how to check this fact when $n = 2$: Write

y_1, y_2 basis y_H

$$y_p = y = u_1y_1 + u_2y_2\,.$$

Thus

set $= 0$

$$y_p' = u_1y_1' + u_2y_2' + (u_1'y_1 + u_2'y_2).$$

Set

$$(u_1'y_1 + u_2'y_2) = 0.$$

Then

set $= f$

$$y'' = u_1y_1'' + u_2y_2'' + (u_1'y_1' + u_2'y_2').$$

Set

$$(u_1'y_1' + u_2'y_2') = f.$$

Notice that the two $(\ldots)$ equations are equivalent to the matrix equation

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

which is equivalent to the $n = 2$ version of the claimed condition for y_p. Under these conditions we compute

DE

$y'' + p_1y' + p_0y = f$

$L(y_p)$:

$$\begin{aligned} &p_0\,[y = u_1y_1 + u_2y_2] \\ +\,&p_1\,[y' = u_1y_1' + u_2y_2'] \\ +\,&1\,[y'' = u_1y_1'' + u_2y_2'' + f] \\ &L(y) = u_1L(y_1) + u_2L(y_2) + f \\ &L(y) = 0 + 0 + f = f \end{aligned}$$

$p(r) = r^2 + 4r - 5 = (r+5)(r-1)$

Exercise 9) Rework Exercise 7a with variation of parameters, i.e. find a particular solution to

$$y'' + 4y' - 5y = \underline{4}\,e^x \qquad y_H = \text{span}\{e^x, e^{-5x}\}$$

of the form

$$y_P(x) = u_1(x)y_1(x) + u_2(x)y_2(x) = u_1 e^x + u_2 e^{-5x}.$$

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

$$\begin{bmatrix} e^x & e^{-5x} \\ e^x & -5e^{-5x} \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 4e^x \end{bmatrix} \qquad A\vec{x} = \vec{b}, \quad \vec{x} = A^{-1}\vec{b}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{-5e^{-4x} - e^{-4x}}\begin{bmatrix} -5e^{-5x} & -e^{-5x} \\ -e^x & e^x \end{bmatrix}\begin{bmatrix} 0 \\ 4e^x \end{bmatrix}$$

$$= -\frac{1}{6}e^{4x}\begin{bmatrix} -4e^{-4x} \\ 4e^{2x} \end{bmatrix}$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 2/3 \\ -2/3\, e^{6x} \end{bmatrix}$$

$$u_1 = \frac{2}{3}x$$

$$u_2 = -\frac{1}{9}e^{6x}$$

$$y_P = u_1 y_1 + u_2 y_2$$

$$= \frac{2}{3}xe^x - \frac{1}{9}e^{6x}e^{-5x} = \frac{2}{3}xe^x \underbrace{- \frac{1}{9}e^x}_{\text{is a } y_H}$$

$$y = y_P + y_H = \frac{2}{3}xe^x \cancel{- \frac{1}{9}e^x} + c_1e^x + c_2e^{-5x}$$

this is the y_P we got use undetermined coef's & factoring

since this is a homogeneous solution we may leave it there or incorporate it into the general homog. soltn

Appendix: The following two theorems justify the method of undetermined coefficients, in both the "base case" and the "extended case." There is also a related homework problem.

Theorem 0:

- Let V and W be vector spaces. Let V have dimension $n < \infty$ and let $\{y_1, y_2, ..., y_n\}$ be a basis for V.
- Let $L : V \to W$ be a linear transformation, i.e. $L(y+z) = L(y) + L(z)$ and $L(c\,y) = c\,L(y)$ holds

$\forall\ y, z \in V, c \in \mathbb{R}$.) Consider the range of L, i.e.

$$Range(L) := \{L(d_1 y_1 + d_2 y_2 + ... + d_n y_n) \in W, \text{such that each } d_j \in \mathbb{R}\}$$
$$= \{d_1 L(y_1) + d_2 L(y_2) + ... + d_n L(y_n) \in W, \text{such that each } d_j \in \mathbb{R}\}$$
$$= span\{L(y_1), L(y_2), ... L(y_n)\}.$$

Then $Range(L)$ is $n - dimensional$ if and only if the only solution to $L(y) = 0$ is $y = 0$.

proof:

(i) $\Leftarrow$: The only solution to $L(y) = 0$ is $y = 0$ implies $Range(L)$ is $n - dimensional$:

If we can show $L(y_1), L(y_2), ... L(y_n)$ are linearly independent, then they will be a basis for $Range(L)$ and thus this subspace will have dimension n. So, consider the dependency equation:

$$d_1 L(y_1) + d_2 L(y_2) + ... + d_n L(y_n) = 0\,.$$

Because L is a linear transformation, we can rewrite this equation as

$$L(d_1 y_1 + d_2 y_2 + ... + d_n y_n) = 0.$$

Under our assumption that the only homogeneous solution is the zero vector, we deduce

$$d_1 y_1 + d_2 y_2 + ... + d_n y_n = 0.$$

Since $y_1, y_2, ..., y_n$ are a basis they are linearly independent, so $d_1 = d_2 = ... = d_n = 0$.

□

(ii) $\Rightarrow$: $Range(L)$ is $n - dimensional$ implies the only solution to $L(y) = 0$ *is* $y = 0$: Since the range of L is $n - dimensional$, $L(y_1), L(y_2), ... L(y_n)$ must be linearly independent. Now, let $y = d_1 y_1 + d_2 y_2 + ... + d_n y_n$ be a homogeneous solution, $L(y) = 0$. In other words,

$$L(d_1 y_1 + d_2 y_2 + ... + d_n y_n) = 0$$
$$\Rightarrow d_1 L(y_1) + d_2 L(y_2) + ... + d_n L(y_n) = 0$$
$$\Rightarrow d_1 = d_2 = ... = d_n = 0 \Rightarrow y = 0\,.$$

□

Theorem 1 Let Let V and W be vector spaces, both with the same dimension $n < \infty$. Let $L : V \to W$ be a linear transformation. Let the only solution to $L(y) = 0$ be $y = 0$. Then for each $f \in W$ there is a unique $y \in V$ with $L(y) = f$.

proof: By Theorem 0, the dimension of $Range(L)$ is $n - dimensional$. Therefore it must be all of W. So for each $f \in W$ there is at least one $y_P \in V$ with $L(y_P) = f$. But the general solution to $L(y) = f$ is $y = v_P + y_H$, where y_H is the general solution to the homogeneous equation. By assumption, $y_H = 0$, so the particular solution is unique.

□

Remark: In the base case of undetermined coefficients, $W = V$. In the extended case, W is the space in which f lies, and $V = x^s W$, i.e. the space of all functions which are obtained from ones in W by multiplying them by x^s. This is because if L factors as

$$L = (D - r_1 I)^{k_1} \circ (D - r_2 I)^{k_2} \circ \ldots \circ (D - r_m I)^{k_m}$$

and if f is in a subspace W associated with the characteristic polynomial root r_m, then for $s = k_m$ the factor $(D - r_m I)^{k_m}$ of L will transform the space $V = x^s W$ back into W, and not transform any non-zero function in V into the zero function. And the other factors of L will then preserve W, also without transforming any non-zero elements to the zero function.

$L : V \to W$ linear

$\dim V = n < \infty$

$\text{nullspace}(L) = \{ y \in V \text{ s.t. } L(y) = 0 \} \subset V$ subspace

dim is called "nullity".

$\text{image}(L) = \{ L(y) \text{ s.t. } y \in V \} \subset W$ subspace

dim is called "rank"

$$\boxed{\text{rank + nullity theorem} \quad \text{rank} + \text{nullity} = \dim(V)}$$

Cor if $\dim V = \dim W \overset{=n}{,}$ and if nullity $= 0$ (nullspace of $L = \{\vec{0}\}$)

$\Rightarrow$ rank $= \dim V$

$\Rightarrow$ image$(L) = W$ (since the only n-dim'l subspace of W is W.)

and if to solve $L(y_p) = f$ for $f \in W$

$\exists!\ y_p \in V$

image$(L) = W$

$y = y_p + y_H$, $y_H = 0$

basis for image$(L) = \{f_1, f_2, .. f_k\}$

$= \{L(y_1), L(y_2), .. L(y_k)\}$

basis for nullspace of L

$= \{z_1, z_2, .. z_m\}$

(aiming to show $k + m = n$)

Claim $\{y_1, y_2, .. y_k, z_1, z_2, \ldots z_m\}$ are a basis for V!

- linear independence:

Let $\cancel{c_1 y_1 + c_2 y_2 + \ldots + c_k y_k} + d_1 z_1 + d_2 z_2 + \ldots + d_m z_m = 0$

take L! $c_1 L(y_1) + c_2 L(y_2) + \ldots + c_k L(y_k) + \cancel{0 + 0 + \ldots + 0} = 0$

$c_1 f_1 + c_2 f_2 + \ldots + c_k f_k = 0 \Longrightarrow c_1 = c_2 = .. c_k = 0$

($\{f_j\}$ linearly ind)

$\Rightarrow d_1 z_1 + \ldots + d_m z_m = 0 \Rightarrow d_1 = d_2 .. = d_m = 0$ ☑

- span: Let $y \in V$. : take L: $L(y) = f = d_1 f_1 + d_2 f_2 + \ldots + d_k f_k$

$= L(d_1 y_1 + d_2 y_2 + \ldots + d_k y_k)$

$\Rightarrow L(y - (d_1 y_1 + d_2 y_2 + \ldots + d_k y_k)) = 0$

$\Rightarrow y - (d_1 y_1 + d_2 y_2 + \dots + d_k y_k)$ is in nullspace

so it $= c_1 z_1 + c_2 z_2 + \dots + c_m z_m$ some c_j's

$\Rightarrow y = c_1 z_1 + \dots + c_m z_m + d_1 y_1 + d_2 y_2 + \dots + d_k y_k$ $\blacksquare$

Math 2280-001

Fri Feb 24 → Finish Wednesday notes §3.5

Fri: finish undetermined coef's (rank + nullity theorem).

variation of parameters

to do these notes Monday

Section 3.6: forced oscillations in mechanical systems (and as we shall see in section 3.7, also in electrical circuits) overview:

We study solutions $x(t)$ to

$$m\,x'' + c\,x' + k\,x = F_0 \cos(\omega\,t)$$

using section 3.5 undetermined coefficients algorithms.

- undamped $(c = 0)$:
  In this case the complementary homogeneous differential equation for $x(t)$ is

$$m\,x'' + k\,x = 0$$

$$x'' + \frac{k}{m}x = 0$$

$$x'' + \omega_0^2\,x = 0$$

which has simple harmonic motion solutions

$$x_H(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) = C_0 \cos(\omega_0 t - \alpha)\,.$$

So for the non-homongeneous DE the section 5.5 method of undetermined coefficients implies we can find particular and general solutions as follows:

- $\omega \neq \omega_0 := \sqrt{\frac{k}{m}} \Rightarrow x_P = A\cos(\omega\,t)$ because only even derivatives, we don't need

sin $(\omega\,t)$ terms !!

$$\Rightarrow x = x_P + x_H = A\cos(\omega\,t) + C_0 \cos(\omega_0 t - \alpha_0)\,.$$

- $\omega \neq \omega_0$ but $\omega \approx \omega_0$, $A \approx C_0$ Beating!
- $\omega = \omega_0$ case 2 section 3.5 undetermined coefficients; since

$$p(r) = r^2 + \omega_0^2 = (r + i\omega_0)^1 (r - i\omega_0)^1$$

our undetermined coefficients guess is

$$x_P = t^1 \left(A\cos(\omega_0\,t) + B\sin(\omega_0\,t)\right)$$

$$\Rightarrow x = x_P + x_H = C\,t\cos(\omega\,t - \alpha) + C_0\cos(\omega_0 t - \alpha_0)\,.$$

("pure" resonance!)

- damped $(c > 0)$: in all cases $x_P = A\cos(\omega\,t) + B\sin(\omega\,t) = C\cos(\omega\,t - \alpha)$ (because the

roots of the characteristic polynomial are never purely imagninary $\pm i\,\omega$ when $c > 0$).

- underdamped: $x = x_P + x_H = C\cos(\omega\,t - \alpha) + e^{-p\,t} C_1 \cos(\omega_1 t - \alpha_1)\,.$
- critically-damped: $x = x_P + x_H = C\cos(\omega\,t - \alpha) + e^{-p\,t}(c_1 t + c_2)\,.$
- over-damped: $x = x_P + x_H = C\cos(\omega\,t - \alpha) + c_1 e^{-r_1 t} + c_2 e^{-r_2 t}\,.$