Math 2280-001
Week 7: 3.5-3.6
Wed Feb 22

Section 3.5: Finding y_P for non-homogeneous linear differential equations
$$L(y) = f$$
(so that you can use the general solution $y = y_P + y_H$ to solve initial value problems, and because
sometimes a good choice for y_P contains the most essential information in dynamical systems problems).

There are two methods we will use:

today
• The method of <u>undetermined coefficients</u> uses guessing algorithms, and works for constant coefficient
linear differential equations with certain classes of functions $f(x)$ for the non-homogeneous term. The
method seems magic, but actually relies on vector space theory. We've already seen simple examples of
this, where we seemed to pick particular solutions out of the air. This method is the main focus of section
3.5.

Friday
• The method of <u>variation of parameters</u> is more general, and yields an integral formula for a particular
solution y_P, assuming you are already in possession of a basis for the homogeneous solution space. This
method has the advantage that it works for any linear differential equation and any (continuous) function f.
It has the disadvantage that the formulas can get computationally messy especially for differential
equations of order $n > 2$.

The easiest way to explain the method of <u>undetermined coefficients</u> is with examples.

*Roughly speaking, you make a "guess" with free parameters (undetermined coefficients) that "looks like"
the right side. AND, you need to include all possible terms in your guess that could arise when you apply
L to the terms you know you want to include.*

We'll make this more precise later in today's notes.

<u>Exercise 1)</u> Find a particular solution $y_P(x)$ for the differential equation
$$L(y) := y'' + 4y' - 5y = \boxed{10x + 3} \quad f_1$$
Hint: try $y_P(x) = d_1 x + d_2$ because L transforms such functions into ones of the form $b_1 x + b_2$. d_1, d_2
are your "undetermined coefficients", for the given right hand side coefficients $b_1 = 10$, $b_2 = 3$.

$$-5 (y_P = d_1 x + d_2)$$
$$+4 (y_P' = d_1)$$
$$+1 (y_P'' = 0)$$
_____
$$L(y_P) = x(-5d_1) + 1(-5d_2 + 4d_1)$$
$$\underset{=}{\text{want}} \quad 10x \qquad\qquad + 3$$

equate coeff's of "x" and "1"

$$-5d_1 = 10 \qquad\qquad -5d_2 + 4d_1 = 3$$
$$\boxed{d_1 = -2} \qquad\qquad \frac{-8}{-5d_2 = 11} \Rightarrow \boxed{d_2 = -\frac{11}{5}}$$

So $\boxed{y_P(x) = -2x - \frac{11}{5}}$

<u>Exercise 2)</u> Use your work in <u>1</u> and your expertise with homogeneous linear differential equations to find the <u>general solution</u> to

$$y'' + 4y' - 5y = \boxed{10x + 3} \quad f_1$$

$$\Downarrow$$

$$y = y_P + y_H$$

$$\Longrightarrow y = -2x - \tfrac{11}{5} + c_1 e^x + c_2 e^{-5x}$$

$$\Downarrow \quad y'' + 4y' - 5y = 0$$
$$p(r) = r^2 + 4r - 5$$
$$= (r-1)(r+5)$$
$$r = 1, -5 \qquad y_H = c_1 e^x + c_2 e^{-5x}$$

<u>Exercise 3)</u> Find a particular solution to

$$L(y) = y'' + 4y' - 5y = \boxed{14\,e^{2x}} \quad f_2$$

Hint: try $y_P = d\,e^{2x}$ because L transforms functions of that form into ones of the form $b\,e^{2x}$, i.e.

$L(d\,e^{2x}) = b\,e^{2x}$. "d" is your "undetermined coefficient" for $b = 14$.

$$-5\,(\;y_P = de^{2x}\;)$$
$$+4\,(\;y_P' = 2de^{2x}\;)$$
$$+1\,(\;y_P'' = 4de^{2x}\;)$$

$$\boxed{y_{P_2}(x) = 2e^{2x}}$$

want

$$\overline{L(y_P) = e^{2x}d\,(-5 + 8 + 4) = \boxed{7}de^{2x} = \boxed{14}e^{2x}}$$

so $\boxed{d = 2}$

<u>Exercise 4a)</u> Use superposition (linearity of the operator L) and your work from the previous exercises to find the general solution to

$$f_2 - 2f_1$$

$$L(y) = y'' + 4y' - 5y = \boxed{14\,e^{2x}} - 20x - 6$$

$L(y_{P_1}) = f_1 = 10x + 3$

$y_{P_1} = -2x - \tfrac{11}{5}$

<u>4b)</u> Solve (or at least set up the problem to solve) the initial value problem

$$y'' + 4y' - 5y = 14\,e^{2x} - 20x - 6$$
$$y(0) = 4$$
$$y'(0) = -4.$$

$L(y_{P_2}) = f_2$

$y_{P_2} = 2e^{2x}$

$\Longrightarrow L(y_{P_2} - 2y_{P_1}) = f_2 - 2f_1$

<u>4c)</u> Check your answer with technology.

because L is linear.

> $with(DEtools):$

> $dsolve(\{y''(x) + 4 \cdot y'(x) - 5 \cdot y(x) = 14 \cdot e^{2 \cdot x} - 20 \cdot x - 6, y(0) = 4, y'(0) = -4\});$

$$y(x) = \frac{8}{5}\,e^{-5x} - 4\,e^x + 2\,e^{2x} + 4x + \frac{22}{5} \qquad\qquad (1)$$

$c_2 \quad\quad c_1 \quad\quad \underbrace{\quad}_{y_P}$

gen'l

$\Longrightarrow$ sol'n to 4a) if

$$y = y_{P_2} - 2y_{P_1} + y_H$$

$$y = \underbrace{2e^{2x} - 2(-2x - \tfrac{11}{5})}_{y_P} + \underbrace{c_1 e^x + c_2 e^{-5x}}_{y_H}$$

IVP $\quad y(0) = 4 = 2 + \tfrac{22}{5} + c_1 + c_2$

$\quad\quad y'(0) = -4 = 4 + 4 + c_1 - 5c_2$

<u>Exercise 5)</u> Find a particular solution to
$$L(y) := y'' + 4y' - 5y = 2\cos(3x).$$
Hint: To solve $L(y) = f$ we hope that f is in some finite dimensional subspace V that is preserved by L, i.e.
$L : V \to V$.

- In <u>Exercise 1</u> $V = span\{1, x\}$ and so we guessed $y_P = d_1 + d_2 x$.

$L : span\{1, x\} \to span\{1, x\}.$

- In <u>Exercise 3</u> $V = span\{e^{2x}\}$ and so we guessed $y_P = d e^{2x}$.

$L : span\{e^{2x}\} \to span\{e^{2x}\}.$

- What's the smallest subspace V we can take in the current exercise? Can you see why
$V = span\{\cos(3x)\}$ and a guess of $y_P = d\cos(3x)$ <u>won't</u> work?

$-5 \left(y_P = d_1 \cos 3x + d_2 \sin 3x \right)$

$+ 4 \left(y_P' = -3d_1 \sin 3x + 3d_2 \cos 3x \right)$

$+ 1 \left(y_P'' = -9d_1 \cos 3x - 9d_2 \sin 3x \right)$

$L : span\{\cos 3x, \sin 3x\} \to span\{\cos 3x, \sin 3x\}.$

$L(y_P) = \cos 3x \left(-5d_1 + 12d_2 - 9d_1 \right)$
$\quad\quad\quad + \sin 3x \left(-5d_2 - 12d_1 - 9d_2 \right)$

want
$= (\cos 3x)(2)$
$+ (\sin 3x)(0)$

$\Rightarrow \quad -14d_1 + 12d_2 = 2$
$\quad\quad -12d_1 - 14d_2 = 0$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\begin{bmatrix} -7 & 6 \\ -6 & -7 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\frac{49}{36}$

$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \frac{1}{85} \begin{bmatrix} -7 & -6 \\ 6 & -7 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{85} \begin{bmatrix} -7 \\ 6 \end{bmatrix}$

$y_P = -\frac{7}{85}\cos 3x + \frac{6}{85}\sin 3x$

All of the previous exercises rely on:

<u>Method of undetermined coefficients</u> (base case): Let $L : V \to V$ be a linear transformation, with V a finite dimensional vector space, and let $f \in V$. Then $\exists! \ y_P \in V$ with $L(y_P) = f$ if and only if the only $y \in V$ for which $L(y) = 0$ is $y = 0$.

<u>**why:**</u> You definitely learned this fact in Math 2270, for the special case of matrix transformations $L : \mathbb{R}^n \to \mathbb{R}^n$ given by $L(\underline{x}) = A_{n \times n}\underline{x}$. (Each non-homogeneous matrix equation $A\underline{x} = \underline{b}$ has a unique solution $\underline{x}$ if and only if A reduces to the identity matrix I, if and only if the only solution to the homogeneous equation $A\underline{x} = \underline{0}$ is $\underline{x} = \underline{0}$.) The theorem above is a generalization of this fact to general linear transformations $L : V \to V$. There is an "appendix" explaining the reasoning at the end of today's notes. In fact, it's a special case of the "rank+nullity" theorem, which you may have learned in Math 2270, and which you will review in a homework problem.
http://en.wikipedia.org/wiki/Rank–nullity_theorem

Exercise 6) Use the method of undetermined coefficients to guess the form for a particular solution $y_P(x)$ for a constant coefficient differential equation
$$L(y) := y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = f$$
In other words, *specify* a finite-dimensional subpace V with $L : V \to V$ and with the only $y \in V$ satisfying $L(y) = 0$ being the zero function $y \equiv 0$.

6a) $L(y) = x^3 + 6x - 5$

$V = \text{span}\{1, x, x^2, x^3\}$
$y_P = d_1 + d_2 x + d_3 x^2 + d_4 x^3$

6b) $L(y) = 4 e^{2x} \sin(3x)$

$V = \text{span}\{e^{2x} \sin 3x, e^{2x} \cos 3x\}$
$y_P = d_1 e^{2x} \sin 3x + d_2 e^{2x} \cos 3x$

6c) $L(y) = x \cos(2x)$

$V = \text{span}\{x \cos 2x, x \sin 2x, \cos 2x, \sin 2x\}$
$y_P = d_1 x \cos 2x + d_2 x \sin 2x + d_3 \cos 2x + d_4 \sin 2x$

BUT LOOK OUT
Exercise 7a) Find a particular solution to
$$y'' + 4y' - 5y = 4 e^x .$$
Hint: since $p(r) = r^2 + 4r - 5 = (r-1)(r+5)$ and $y_H = c_1 e^x + c_2 e^{-5x}$, a guess of $y_P = a e^x$ will <u>not</u> work (and $span\{e^x\}$ does not satisfy the "base case" conditions for undetermined coefficients). Instead try
$$y_P = d \, x \, e^x$$
and factor $L = D^2 + 4D - 5I = (D + 5I) \circ (D - I)$.

7b) check work with technology

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> with(DEtools) :
> dsolve(y''(x) + 4·y'(x) − 5·y(x) = 4·e^x, y(x));
>
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$y_P = A e^x \qquad V = \text{span}\{e^x\}$
$(y_H(x) = c_1 e^x + c_2 e^{-5x})$
$L(A e^x) = 0 \qquad \underline{\text{oops}}.$

modify $\qquad V = \text{span}\{x e^x\}$.

$\underbrace{[D+5I] \circ [D-I](dx e^x)}_{d e^x}$

$= [D+5I](d e^x)$
$= d e^x + 5 d e^x = 6 d e^x$
$\underset{\text{want}}{=} 4 e^x$

$[D - aI] \, f(x) e^{ax} = f'(x) e^{ax}$
$(f'(x) e^{ax} + f(x) a e^{ax} - a f(x) e^{ax})$
$\text{span}\{x e^x\}$
$L : V \longrightarrow W$
$\text{span}\{e^x\}$

$d = \frac{2}{3} \qquad \boxed{y_P = \frac{2}{3} x e^x}$

A vector space theorem like the one for the base case, except for $L : V \rightarrow W$, combined with our understanding of how to factor constant coefficient differential operators (as in last week's homework) leads to an extension of the method of undetermined coefficients, for right hand sides which can be written as sums of functions having the indicated forms below. See the discussion in section 3.5 of the text, pages 190-191 of the new edition of our text, and the table 3.5.1, reproduced here.

<u>Method of undetermined coefficients (extended case)</u>: If L has a factor $(D - r I)^s$ and $e^{r x}$ is also associated with (a portion of) the right hand side $f(x)$ then the corresponding guesses you would have made in the "base case" need to be multiplied by x^s, as in <u>Exercise 7</u>. (If you understood the homework problem last week about factoring L into composition of terms like $(D - r I)^s$, then you have an inkling of why this recipe works. If you didn't understand that last week problem, there's another one this week so you get a second chance. :-)) You may also need to use superposition, as in <u>Exercise 4</u>, if different portions of $f(x)$ are associated with different exponential functions.

Extended case of undetermined coefficients

$f(x)$	y_P	$s > 0$ when $p(r)$ has these roots:
$P_m(x) = b_0 + b_1 + ... + b_m x^m$	$x^s \left(c_0 + c_1 x + c_2 x^2 + ... + c_m x^m \right)$	$r = 0$
$b_1 \cos(\omega x) + b_2 \sin(\omega x)$	$x^s \left(c_1 \cos(\omega x) + c_2 \sin(\omega x) \right)$	$r = \pm i \omega$
$e^{a x} \left(b_1 \cos(\omega x) + b_2 \sin(\omega x) \right)$	$x^s e^{a x} \left(c_1 \cos(\omega x) + c_2 \sin(\omega x) \right)$	$r = a \pm i \omega$
$b_0 e^{a x}$	$x^s c_0 e^{a x}$	$r = a$
$\left(b_0 + b_1 + ... + b_m x^m \right) e^{a x}$	$x^s \left(c_0 + c_1 x + c_2 x^2 + ... + c_m x^m \right) e^{a x}$	$r = a$

Exercise 8) Set up the undetermined coefficients particular solutions for the examples below. When necessary use the extended case to modify the undetermined coefficients form for y_P. Use technology to check if your "guess" form was right.

$$L(y) := y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = f$$

Y_H contains $e^{0x} = 1$
$x e^{0x} = x$

8a) $y''' + 2 y'' = \boxed{x^2 + 6 x}$
(So the characteristic polynomial for $L(y) = 0$ is $r^3 + 2 r^2 = r^2(r+2) = (r - 0)^{\textcircled{2}}(r + 2)$.)

> $dsolve(y'''(x) + 2 \cdot y''(x) = x^2 + 6 \cdot x, y(x))$;

$$y(x) = \frac{1}{24} x^4 + \frac{5}{12} x^3 + \frac{1}{4} e^{-2x} \_C1 - \frac{5}{8} x^2 + \_C2 x + \_C3 \qquad (2)$$

$$Y_p = x^{\textcircled{2}}\left(d_1 x^2 + d_2 x + d_3\right) = d_1 x^4 + d_2 x^3 + d_3 x^2$$

$$L = D^3 + 2 D^2$$
$$[D+2] \circ [D \cdot D]$$

8b) $y'' - 4 y' + 13 y = 4 e^{2x} \sin(3 x)$ ❓
(So the characteristic polynomial for $L(y) = 0$ is
$r^2 - 4 r + 13) = (r - 2)^2 + 9 = (r - 2 + 3 i)(r - 2 - 3 i)$.)

> $dsolve(y''(x) - 4 \cdot y'(x) + 13 \cdot y(x) = 4 \cdot e^{2 \cdot x} \cdot \sin(3 \cdot x), y(x))$;

$$y(x) = e^{2x} \sin(3 x) \_C2 + e^{2x} \cos(3 x) \_C1 - \frac{2}{3} e^{2x} \cos(3 x) x \qquad (3)$$

which f lies, and $V = x^s W$, i.e. the space of all functions which are obtained from ones in W by multiplying them by x^s. This is because if L factors as

$$L = (D - r_1 I)^{k_1} \circ (D - r_2 I)^{k_2} \circ \dots \circ (D - r_m I)^{k_m}$$

and if f is in a subspace W associated with the characteristic polynomial root r_m, then for $s = k_m$ the factor $(D - r_m I)^{k_m}$ of L will transform the space $V = x^s W$ back into W, and not transform any non-zero function in V into the zero function. And the other factors of L will then preserve W, also without transforming any non-zero elements to the zero function.

$$L : V \longrightarrow W \qquad \text{linear}$$
$$\uparrow$$
$$\dim V = n < \infty$$

$$\text{nullspace}(L) = \{ y \in V \text{ s.t } L(y) = 0 \} \subset V \qquad \text{subspace}$$
$$\text{dim is called "nullity".}$$

$$\text{image}(L) = \{ L(y) \text{ s.t } y \in V \} \subset W \qquad \text{subspace}$$
$$\text{dim is called "rank."}$$

rank + nullity theorem
$$\text{rank + nullity} = \dim(V)$$

$$\underline{\text{Cor}} \quad \text{if } \dim V = \dim W \overset{= n}{,} \text{ and if nullity} = 0$$
$$\qquad \qquad \qquad \qquad (\text{nullspace of } L = \{\vec{0}\})$$
$$\Rightarrow \text{rank} = \dim V$$
$$\Rightarrow \text{image}(L) = W$$
$$\text{and if to solve } L(y_p) = f \quad \text{for } f \in W$$
$$\exists! \ y_p \in V$$
$$\uparrow$$
$$\text{image}(L) = W$$
$$y = y_p + \underbrace{y_H}_{=0}$$