

Section 3.5: Finding  $y_p$  for non-homogeneous linear differential equations

$$L(y) = f$$

(so that you can use the general solution  $y = y_p + y_H$  to solve initial value problems, and because sometimes a good choice for  $y_p$  contains the most essential information in dynamical systems problems).

There are two methods we will use:

• The method of undetermined coefficients <sup>today</sup> uses guessing algorithms, and works for constant coefficient linear differential equations with certain classes of functions  $f(x)$  for the non-homogeneous term. The method seems magic, but actually relies on vector space theory. We've already seen simple examples of this, where we seemed to pick particular solutions out of the air. This method is the main focus of section 3.5.

• The method of variation of parameters <sup>Friday</sup> is more general, and yields an integral formula for a particular solution  $y_p$ , assuming you are already in possession of a basis for the homogeneous solution space. This method has the advantage that it works for any linear differential equation and any (continuous) function  $f$ . It has the disadvantage that the formulas can get computationally messy especially for differential equations of order  $n > 2$ .

The easiest way to explain the method of undetermined coefficients is with examples.

*Roughly speaking, you make a "guess" with free parameters (undetermined coefficients) that "looks like" the right side. AND, you need to include all possible terms in your guess that could arise when you apply  $L$  to the terms you know you want to include.*

We'll make this more precise later in today's notes.

Exercise 1) Find a particular solution  $y_p(x)$  for the differential equation  $f_1$

$$L(y) := y'' + 4y' - 5y = \boxed{10x + 3}$$

Hint: try  $y_p(x) = d_1x + d_2$  because  $L$  transforms such functions into ones of the form  $b_1x + b_2$ .  $d_1, d_2$  are your "undetermined coefficients", for the given right hand side coefficients  $b_1 = 10, b_2 = 3$ .

$$\begin{aligned} & -5(y_p = d_1x + d_2) \\ & + 4(y_p' = d_1) \\ & + 1(y_p'' = 0) \end{aligned}$$

$$L(y_p) = x(-5d_1) + 1(-5d_2 + 4d_1)$$

want  $10x + 3$

equate coeff's of "x" and "1"

$$\begin{aligned} -5d_1 &= 10 & -5d_2 + 4d_1 &= 3 \\ \boxed{d_1 = -2} & & -5d_2 - 8 &= 3 \\ & & -5d_2 &= 11 \Rightarrow \boxed{d_2 = -\frac{11}{5}} \end{aligned}$$

So  $\boxed{y_p(x) = -2x - \frac{11}{5}}$

Exercise 2) Use your work in 1 and your expertise with homogeneous linear differential equations to find the general solution to

$$y'' + 4y' - 5y = \boxed{10x + 3} \quad f_1$$

$$y = y_p + y_H$$

$$y'' + 4y' - 5y = 0$$

$$p(r) = r^2 + 4r - 5$$

$$= (r - 1)(r + 5)$$

$$r = 1, -5$$

$$y_H = c_1 e^x + c_2 e^{-5x}$$

$$\Rightarrow y = -2x - \frac{11}{5} + c_1 e^x + c_2 e^{-5x}$$

Exercise 3) Find a particular solution to

$$L(y) = y'' + 4y' - 5y = \boxed{14e^{2x}} \quad f_2$$

Hint: try  $y_p = d e^{2x}$  because  $L$  transforms functions of that form into ones of the form  $b e^{2x}$ , i.e.

$L(d e^{2x}) = b e^{2x}$ . " $d$ " is your "undetermined coefficient" for  $b = 14$ .

$$-5 (y_p = d e^{2x})$$

$$+ 4 (y_p' = 2d e^{2x})$$

$$+ 1 (y_p'' = 4d e^{2x})$$

$$\underline{\quad\quad\quad} \quad L(y_p) = e^{2x} d (-5 + 8 + 4) = \boxed{7d} e^{2x} = \boxed{14} e^{2x}$$

want

$$\text{so } \boxed{d=2}$$

Exercise 4a) Use superposition (linearity of the operator  $L$ ) and your work from the previous exercises to find the general solution to

$$L(y) = y'' + 4y' - 5y = \boxed{14e^{2x}} - \boxed{20x - 6} \quad f_2 - 2f_1$$

4b) Solve (or at least set up the problem to solve) the initial value problem

$$y'' + 4y' - 5y = 14e^{2x} - 20x - 6$$

$$y(0) = 4$$

$$y'(0) = -4$$

4c) Check your answer with technology.

$$L(y_{p1}) = f_1 = 10x + 3$$

$$y_{p1} = -2x - \frac{11}{5}$$

$$L(y_{p2}) = f_2$$

$$y_{p2} = 2e^{2x}$$

$$\Rightarrow L(y_{p2} - 2y_{p1}) = f_2 - 2f_1$$

because  $L$  is linear.

> with (DEtools) :

> dsolve( $\{y''(x) + 4y'(x) - 5y(x) = 14 \cdot e^{2x} - 20 \cdot x - 6, y(0) = 4, y'(0) = -4\}$ );

$$y(x) = \frac{8}{5} e^{-5x} - 4e^x + \boxed{2e^{2x}} + \boxed{4x} + \boxed{\frac{22}{5}}$$

(1)

$\Rightarrow$  gen'l sol'n to 4a) if

$$y = y_{p2} - 2y_{p1} + y_H$$

$$y = \underbrace{2e^{2x} - 2(-2x - \frac{11}{5})}_{y_p} + \underbrace{c_1 e^x + c_2 e^{-5x}}_{y_H}$$

$$\text{IVP } y(0) = 4 = 2 + \frac{22}{5} + c_1 + c_2$$

$$y'(0) = -4 = 4 + 4 + c_1 - 5c_2$$

Exercise 5) Find a particular solution to

$$L(y) := y'' + 4y' - 5y = 2 \cos(3x).$$

Hint: To solve  $L(y) = f$  we hope that  $f$  is in some finite dimensional subspace  $V$  that is preserved by  $L$ , i.e.  $L : V \rightarrow V$ .

• In Exercise 1  $V = \text{span}\{1, x\}$  and so we guessed  $y_p = d_1 + d_2 x$ .

$$L : \text{span}\{1, x\} \rightarrow \text{span}\{1, x\}.$$

• In Exercise 3  $V = \text{span}\{e^{2x}\}$  and so we guessed  $y_p = d e^{2x}$ .

$$L : \text{span}\{e^{2x}\} \rightarrow \text{span}\{e^{2x}\}.$$

• What's the smallest subspace  $V$  we can take in the current exercise? Can you see why  $V = \text{span}\{\cos(3x)\}$  and a guess of  $y_p = d \cos(3x)$  won't work?

$$-5(y_p = d_1 \cos 3x + d_2 \sin 3x)$$

$$L : \text{span}\{\cos 3x, \sin 3x\} \rightarrow \text{span}\{\cos 3x, \sin 3x\}$$

$$+ 4(y_p' = -3d_1 \sin 3x + 3d_2 \cos 3x)$$

$$+ 1(y_p'' = -9d_1 \cos 3x - 9d_2 \sin 3x)$$

$$\begin{aligned} L(y_p) &= \cos 3x (-5d_1 + 12d_2 - 9d_1) \\ &\quad + \sin 3x (-5d_2 - 12d_1 - 9d_2) \end{aligned}$$

$$\begin{aligned} &\text{want} \\ &= (\cos 3x)(2) \\ &\quad + (\sin 3x)(0) \end{aligned}$$

$$\Rightarrow \begin{aligned} -14d_1 + 12d_2 &= 2 \\ -12d_1 - 14d_2 &= 0 \end{aligned}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{bmatrix} -7 & 6 \\ -6 & -7 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \frac{1}{85} \begin{bmatrix} -7 & -6 \\ 6 & -7 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{85} \begin{bmatrix} -7 \\ 6 \end{bmatrix}$$

$$y_p = -\frac{7}{85} \cos 3x + \frac{6}{85} \sin 3x$$

All of the previous exercises rely on:

Method of undetermined coefficients (base case): Let  $L : V \rightarrow V$  be a linear transformation, with  $V$  a finite dimensional vector space, and let  $f \in V$ . Then  $\exists! y_p \in V$  with  $L(y_p) = f$  if and only if the only  $y \in V$  for which  $L(y) = 0$  is  $y = 0$ .

**why:** You definitely learned this fact in Math 2270, for the special case of matrix transformations  $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$  given by  $L(\mathbf{x}) = A_n \times_n \mathbf{x}$ . (Each non-homogeneous matrix equation  $A \mathbf{x} = \mathbf{b}$  has a unique solution  $\mathbf{x}$  if and only if  $A$  reduces to the identity matrix  $I$ , if and only if the only solution to the homogeneous equation  $A \mathbf{x} = \mathbf{0}$  is  $\mathbf{x} = \mathbf{0}$ .) The theorem above is a generalization of this fact to general linear transformations  $L : V \rightarrow V$ . There is an "appendix" explaining the reasoning at the end of today's notes. In fact, it's a special case of the "rank+nullity" theorem, which you may have learned in Math 2270, and which you will review in a homework problem.

[http://en.wikipedia.org/wiki/Rank-nullity\\_theorem](http://en.wikipedia.org/wiki/Rank-nullity_theorem)

Exercise 6) Use the method of undetermined coefficients to guess the form for a particular solution  $y_p(x)$  for a constant coefficient differential equation

$$L(y) := y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1y' + a_0y = f$$

In other words, specify a finite-dimensional subspace  $V$  with  $L: V \rightarrow V$  and with the only  $y \in V$  satisfying  $L(y) = 0$  being the zero function  $y \equiv 0$ .

6a)  $L(y) = x^3 + 6x - 5$

$$V = \text{span}\{1, x, x^2, x^3\} = W$$

$$y_p = d_1 + d_2x + d_3x^2 + d_4x^3$$

$$L: V \rightarrow W$$

$L(y_p) = f$  will have unique  $y_p \in V$  as long as nullspace of  $L$  is  $\{0\}$ .

6b)  $L(y) = 4e^{2x}\sin(3x)$

$$V = \text{span}\{e^{2x}\sin 3x, e^{2x}\cos 3x\} = W$$

$$y_p = d_1e^{2x}\sin 3x + d_2e^{2x}\cos 3x$$

$$L: V \rightarrow W$$

6c)  $L(y) = x \cos(2x)$

$$V = \text{span}\{x \cos 2x, x \sin 2x, \cos 2x, \sin 2x\}$$

$$y_p = d_1x \cos 2x + d_2x \sin 2x + d_3 \cos 2x + d_4 \sin 2x$$

$$L: V \rightarrow W$$

BUT LOOK OUT

Exercise 7a) Find a particular solution to

$$y'' + 4y' - 5y = 4e^x$$

Hint: since  $p(r) = r^2 + 4r - 5 = (r-1)(r+5)$  and  $y_H = c_1e^x + c_2e^{-5x}$ , a guess of  $y_p = ae^x$  will not work (and  $\text{span}\{e^x\}$  does not satisfy the "base case" conditions for undetermined coefficients). Instead try

$$y_p = dx e^x$$

and factor  $L = D^2 + 4D - 5I = (D+5I) \circ (D-I)$ .

7b) check work with technology

[> with(DEtools):  
 [> dsolve(y''(x) + 4\*y'(x) - 5\*y(x) = 4\*e^x, y(x));  
 [>

modify

$$V = \text{span}\{xe^x\} \xrightarrow{L} \text{span}\{e^x\} = W$$

$$[D+5I] \circ [D-I](dx e^x)$$

$$= [D+5I](de^x)$$

$$= de^x + 5de^x = 6de^x$$

$$\underline{\underline{= 4e^x}}$$

$$d = 2/3$$

$$y_p = Ae^x \quad V = \text{span}\{e^x\}$$

$$(y_H(x) = c_1e^x + c_2e^{-5x})$$

$$L(Ae^x) = 0 \quad \underline{\underline{\text{oops}}}$$

$$[D-aI]f(x)e^{ax} = f'(x)e^{ax}$$

$$\left( f'(x)e^{ax} + f(x)ae^{ax} - af(x)e^{ax} \right)$$

$$\text{span}\{xe^x\}$$

$$L: V \rightarrow W$$

$$\text{span}\{e^x\}$$

$$\boxed{y_p = \frac{2}{3}xe^x}$$

A vector space theorem like the one for the base case, except for  $L : V \rightarrow W$ , combined with our understanding of how to factor constant coefficient differential operators (as in last week's homework) leads to an extension of the method of undetermined coefficients, for right hand sides which can be written as sums of functions having the indicated forms below. See the discussion in section 3.5 of the text, pages 190-191 of the new edition of our text, and the table 3.5.1, reproduced here.

Method of undetermined coefficients (extended case): If  $L$  has a factor  $(D - r I)^s$  and  $e^{rx}$  is also associated with (a portion of) the right hand side  $f(x)$  then the corresponding guesses you would have made in the "base case" need to be multiplied by  $x^s$ , as in Exercise 7. (If you understood the homework problem last week about factoring  $L$  into composition of terms like  $(D - r I)^s$ , then you have an inkling of why this recipe works. If you didn't understand that last week problem, there's another one this week so you get a second chance. :- ) You may also need to use superposition, as in Exercise 4, if different portions of  $f(x)$  are associated with different exponential functions.

Extended case of undetermined coefficients

$f(x)$	$y_p$	$s > 0$ when $p(r)$ has these roots:
$P_m(x) = b_0 + b_1x + \dots + b_mx^m$	$x^s(c_0 + c_1x + c_2x^2 + \dots + c_mx^m)$	$r = 0$
$b_1 \cos(\omega x) + b_2 \sin(\omega x)$	$x^s(c_1 \cos(\omega x) + c_2 \sin(\omega x))$	$r = \pm i\omega$
$e^{ax}(b_1 \cos(\omega x) + b_2 \sin(\omega x))$	$x^s e^{ax}(c_1 \cos(\omega x) + c_2 \sin(\omega x))$	$r = a \pm i\omega$
$b_0 e^{ax}$	$x^s c_0 e^{ax}$	$r = a$
$(b_0 + b_1x + \dots + b_mx^m)e^{ax}$	$x^s(c_0 + c_1x + c_2x^2 + \dots + c_mx^m)e^{ax}$	$r = a$

Exercise 8) Set up the undetermined coefficients particular solutions for the examples below. When necessary use the extended case to modify the undetermined coefficients form for  $y_p$ . Use technology to check if your "guess" form was right.

$$L(y) := y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1y' + a_0y = f$$

8a)  $y''' + 2y'' = x^2 + 6x$

(So the characteristic polynomial for  $L(y) = 0$  is  $r^3 + 2r^2 = r^2(r+2) = (r-0)^2(r+2)$ .)

>  $\text{dsolve}(y'''(x) + 2 \cdot y''(x) = x^2 + 6 \cdot x, y(x));$

$$y(x) = \frac{1}{24}x^4 + \frac{5}{12}x^3 + \frac{1}{4}e^{-2x} - \frac{5}{8}x^2 + C_2x + C_3$$

(2)

$$y_p = x^2(d_1x^2 + d_2x + d_3) = d_1x^4 + d_2x^3 + d_3x^2$$

$$L: V \rightarrow W = \text{span}\{1, x, x^2\}$$

$$\text{span}\{x^2, x^3, x^4\}$$

$$L = D^3 + 2D^2$$

$$[D+2] \circ [D] \circ [D]$$

8b)  $y'' - 4y' + 13y = 4e^{2x}\sin(3x)$

(So the characteristic polynomial for  $L(y) = 0$  is

$$r^2 - 4r + 13 = (r-2)^2 + 9 = (r-2+3i)(r-2-3i). \quad r = 2 \pm 3i$$

>  $\text{dsolve}(y''(x) - 4 \cdot y'(x) + 13 \cdot y(x) = 4 \cdot e^{2 \cdot x} \cdot \sin(3 \cdot x), y(x));$

$$y(x) = e^{2x}\sin(3x) - \frac{2}{3}e^{2x}\cos(3x)x$$

(3)

$$y_H = c_1 e^{2x}\sin 3x + c_2 e^{2x}\cos 3x$$

(for  $y'' - 4y' + 13y = 0$ ).

$$y_p = x^1(d_1 e^{2x}\sin 3x + d_2 e^{2x}\cos 3x)$$

$$\text{span}\{x e^{2x}\sin 3x, x e^{2x}\cos 3x\} \xrightarrow{L} W = \text{span}\{e^{2x}\sin 3x, e^{2x}\cos 3x\}$$

$$L = D^2 - 4D + 13I = [D - (2-3i)I] \circ [D - (2+3i)I]$$

$$L(x e^{(2+3i)x})$$

$$= [D - (2-3i)I] e^{(2+3i)x}$$

$$= (2+3i)e^{(2+3i)x} - (2-3i)e^{(2+3i)x}$$

$$\text{So, } L(xe^{(2+3i)x}) = 6ie^{(2+3i)x}$$

expand each side into real & imaginary fns, and use linearity of  $L$ :

$$\begin{aligned} L(xe^{2x}\cos 3x) + iL(xe^{2x}\sin 3x) \\ = 6i(e^{2x}\cos 3x + ie^{2x}\sin 3x) \end{aligned}$$

equating real fns:

$$L(xe^{2x}\cos 3x) = -6e^{2x}\sin 3x$$

$$L(xe^{2x}\sin 3x) = 6e^{2x}\cos 3x.$$

$$\text{So for } L(y_p) = 4e^{2x}\sin 3x$$

$$\text{we may choose } \boxed{y_p = -\frac{2}{3}xe^{2x}\cos 3x}$$

Not every right hand side is amenable to finding particular solutions via undetermined coefficients. Luckily there is a more general (but technically messier) way that will always work:

Variation of Parameters: The advantage of this method is that it always provides a particular solution, even for non-homogeneous problems in which the right-hand side doesn't fit into a nice finite dimensional subspace preserved by  $L$ , and even if the linear operator  $L$  is not constant-coefficient. The formula for the particular solutions can be somewhat messy to work with, however, once you start computing.

Here's the formula: Let  $y_1(x), y_2(x), \dots, y_n(x)$  be a basis of solutions to the homogeneous DE

$$L(y) := y^{(n)} + p_{n-1}(x)y^{(n-1)} + \dots + p_1(x)y' + p_0(x)y = 0.$$

Then  $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) + \dots + u_n(x)y_n(x)$  is a particular solution to

$$L(y) = f$$

provided the coefficient functions (aka "varying parameters")  $u_1(x), u_2(x), \dots, u_n(x)$  have derivatives satisfying the Wronskian matrix equation

*n=2 case*

$$\begin{bmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \dots & \dots & \dots & \dots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ f \end{bmatrix}$$

Here's how to check this fact when  $n = 2$ : Write

$$y_p = y = u_1 y_1 + u_2 y_2.$$

Thus

*set = 0*

$$y_p' = u_1 y_1' + u_2 y_2' + \underbrace{(u_1' y_1 + u_2' y_2)}_{\text{set} = 0}.$$

Set

$$(u_1' y_1 + u_2' y_2) = 0.$$

Then

*set = f*

$$y_p'' = u_1 y_1'' + u_2 y_2'' + \underbrace{(u_1' y_1' + u_2' y_2')}_{\text{set} = f}.$$

Set

$$(u_1' y_1' + u_2' y_2') = f.$$

Notice that the two ( ...) equations are equivalent to the matrix equation

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

which is equivalent to the  $n = 2$  version of the claimed condition for  $y_p$ . Under these conditions we compute

*DE*

$$y'' + p_1 y' + p_0 y = f$$

*L(y<sub>p</sub>):*

$$\begin{aligned} & p_0 [y = u_1 y_1 + u_2 y_2] \\ & + p_1 [y' = u_1 y_1' + u_2 y_2'] \\ & + 1 [y'' = u_1 y_1'' + u_2 y_2'' + f] \\ & L(y) = u_1 L(y_1) + u_2 L(y_2) + f \\ & L(y) = 0 + 0 + f = f \end{aligned}$$



$$p(r) = r^2 + 4r - 5 = (r+5)(r-1)$$

Exercise 9) Rework Exercise 7a with variation of parameters, i.e. find a particular solution to

$$y'' + 4y' - 5y = 4e^x$$

$$y_H = \text{span}\{e^x, e^{-5x}\}$$

of the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) = u_1e^x + u_2e^{-5x}.$$

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

$$\begin{bmatrix} e^x & e^{-5x} \\ e^x & -5e^{-5x} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 4e^x \end{bmatrix}$$

$$A\vec{x} = \vec{b}$$

$$\vec{x} = A^{-1}\vec{b}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{-5e^{-4x} - e^{-4x}} \begin{bmatrix} -5e^{-5x} & -e^{-5x} \\ -e^x & e^x \end{bmatrix} \begin{bmatrix} 0 \\ 4e^x \end{bmatrix}$$

$$= -\frac{1}{6}e^{4x} \begin{bmatrix} -4e^{-4x} \\ 4e^{2x} \end{bmatrix}$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 2/3 \\ -2/3 e^{6x} \end{bmatrix}$$

$$u_1 = \frac{2}{3}x$$

$$u_2 = -\frac{1}{9}e^{6x}$$

$$y_p = u_1y_1 + u_2y_2$$

$$= \frac{2}{3}xe^x - \frac{1}{9}e^{6x}e^{-5x} = \frac{2}{3}xe^x - \frac{1}{9}e^x$$

is a  $y_H$

$$y = y_p + y_H = \frac{2}{3}xe^x - \cancel{\frac{1}{9}e^x} + c_1e^x + c_2e^{-5x}$$

this is  
the  $y_p$  we  
got use  
undetermined  
coef's & factoring  
L

since this is  
a homogeneous  
solution we  
may leave it  
there or incorporate  
it into the  
general  
homog. soln

Appendix: The following two theorems justify the method of undetermined coefficients, in both the "base case" and the "extended case." There is also a related homework problem.

Theorem 0:

- Let  $V$  and  $W$  be vector spaces. Let  $V$  have dimension  $n < \infty$  and let  $\{y_1, y_2, \dots, y_n\}$  be a basis for  $V$ .
- Let  $L : V \rightarrow W$  be a linear transformation, i.e.  $L(y + z) = L(y) + L(z)$  and  $L(cy) = cL(y)$  holds  $\forall y, z \in V, c \in \mathbb{R}$ .) Consider the range of  $L$ , i.e.

$$\begin{aligned} \text{Range}(L) &:= \{L(d_1 y_1 + d_2 y_2 + \dots + d_n y_n) \in W, \text{ such that each } d_j \in \mathbb{R}\} \\ &= \{d_1 L(y_1) + d_2 L(y_2) + \dots + d_n L(y_n) \in W, \text{ such that each } d_j \in \mathbb{R}\} \\ &= \text{span}\{L(y_1), L(y_2), \dots, L(y_n)\}. \end{aligned}$$

Then  $\text{Range}(L)$  is  $n - \text{dimensional}$  if and only if the only solution to  $L(y) = 0$  is  $y = 0$ .

proof:

(i)  $\Leftarrow$ : The only solution to  $L(y) = 0$  is  $y = 0$  implies  $\text{Range}(L)$  is  $n - \text{dimensional}$ :

If we can show  $L(y_1), L(y_2), \dots, L(y_n)$  are linearly independent, then they will be a basis for  $\text{Range}(L)$  and thus this subspace will have dimension  $n$ . So, consider the dependency equation:

$$d_1 L(y_1) + d_2 L(y_2) + \dots + d_n L(y_n) = 0.$$

Because  $L$  is a linear transformation, we can rewrite this equation as

$$L(d_1 y_1 + d_2 y_2 + \dots + d_n y_n) = 0.$$

Under our assumption that the only homogeneous solution is the zero vector, we deduce

$$d_1 y_1 + d_2 y_2 + \dots + d_n y_n = 0.$$

Since  $y_1, y_2, \dots, y_n$  are a basis they are linearly independent, so  $d_1 = d_2 = \dots = d_n = 0$ . □

(ii)  $\Rightarrow$ :  $\text{Range}(L)$  is  $n - \text{dimensional}$  implies the only solution to  $L(y) = 0$  is  $y = 0$ : Since the range of  $L$  is  $n - \text{dimensional}$ ,  $L(y_1), L(y_2), \dots, L(y_n)$  must be linearly independent. Now, let  $y = d_1 y_1 + d_2 y_2 + \dots + d_n y_n$  be a homogeneous solution,  $L(y) = 0$ . In other words,

$$\begin{aligned} L(d_1 y_1 + d_2 y_2 + \dots + d_n y_n) &= 0 \\ \Rightarrow d_1 L(y_1) + d_2 L(y_2) + \dots + d_n L(y_n) &= 0 \\ \Rightarrow d_1 = d_2 = \dots = d_n = 0 &\Rightarrow y = 0. \end{aligned}$$

□

Theorem 1 Let  $V$  and  $W$  be vector spaces, both with the same dimension  $n < \infty$ . Let  $L : V \rightarrow W$  be a linear transformation. Let the only solution to  $L(y) = 0$  be  $y = 0$ . Then for each  $f \in W$  there is a unique  $y \in V$  with  $L(y) = f$ .

proof: By Theorem 0, the dimension of  $\text{Range}(L)$  is  $n - \text{dimensional}$ . Therefore it must be all of  $W$ . So for each  $f \in W$  there is at least one  $y_p \in V$  with  $L(y_p) = f$ . But the general solution to  $L(y) = f$  is  $y = y_p + y_H$  where  $y_H$  is the general solution to the homogeneous equation. By assumption,  $y_H = 0$ , so the particular solution is unique. □

Remark: In the base case of undetermined coefficients,  $W = V$ . In the extended case,  $W$  is the space in

which  $f$  lies, and  $V = x^s W$ , i.e. the space of all functions which are obtained from ones in  $W$  by multiplying them by  $x^s$ . This is because if  $L$  factors as

$$L = (D - r_1 I)^{k_1} \circ (D - r_2 I)^{k_2} \circ \dots \circ (D - r_m I)^{k_m}$$

and if  $f$  is in a subspace  $W$  associated with the characteristic polynomial root  $r_m$ , then for  $s = k_m$  the factor  $(D - r_m I)^{k_m}$  of  $L$  will transform the space  $V = x^s W$  back into  $W$ , and not transform any non-zero function in  $V$  into the zero function. And the other factors of  $L$  will then preserve  $W$ , also without transforming any non-zero elements to the zero function.

$$L: V \rightarrow W \quad \text{linear}$$

$$\uparrow \\ \dim V = n < \infty$$

$$\text{nullspace}(L) = \{y \in V \text{ s.t. } L(y) = 0\} \subset V \quad \text{subspace}$$

dim is called "nullity".

$$\text{image}(L) = \{L(y) \text{ s.t. } y \in V\} \subset W \quad \text{subspace}$$

dim is called "rank"

rank + nullity theorem

$$\text{rank} + \text{nullity} = \dim(V)$$

Cor if  $\dim V = \dim W = n$  and if nullity = 0  
(nullspace of  $L = \{\vec{0}\}$ )

$$\Rightarrow \text{rank} = \dim V$$

$$\Rightarrow \text{image}(L) = W \quad (\text{since the only } n\text{-dim'l subspace of } W \text{ is } W.)$$

$$\text{and if to solve } L(y_p) = f \quad \text{for } f \in W$$

$$\exists! y_p \in V$$

$$\uparrow \uparrow \\ \text{image}(L) = W \\ y = y_p + \underbrace{y_h}_{=0}$$

$$\begin{aligned} \text{basis for image}(L) &= \{f_1, f_2, \dots, f_k\} \\ &= \{L(y_1), L(y_2), \dots, L(y_k)\} \end{aligned}$$

$$\begin{aligned} \text{basis for nullspace of } L &= \{z_1, z_2, \dots, z_m\} \\ & \quad (\text{aiming to show } k+m=n) \end{aligned}$$

Claim  $\{y_1, y_2, \dots, y_k, z_1, z_2, \dots, z_m\}$  are a basis for  $V$ !

• linear independence:

$$\text{Let } c_1 y_1 + c_2 y_2 + \dots + c_k y_k + d_1 z_1 + d_2 z_2 + \dots + d_m z_m = 0$$

$$\text{take } L! \quad c_1 L(y_1) + c_2 L(y_2) + \dots + c_k L(y_k) + \underbrace{0 + \dots + 0}_{=0} = 0$$

$$c_1 f_1 + c_2 f_2 + \dots + c_k f_k = 0 \implies c_1 = c_2 = \dots = c_k = 0$$

( $\{f_i\}$  linearly ind)

$$\implies d_1 z_1 + \dots + d_m z_m = 0 \implies d_1 = d_2 = \dots = d_m = 0 \quad \square$$

$$\begin{aligned} \bullet \text{span: Let } y \in V. & \text{ take } L: L(y) = f = d_1 f_1 + d_2 f_2 + \dots + d_k f_k \\ & = L(d_1 y_1 + d_2 y_2 + \dots + d_k y_k) \\ \implies L(y - (d_1 y_1 + d_2 y_2 + \dots + d_k y_k)) &= 0 \end{aligned}$$

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$\Rightarrow y = (d_1 y_1 + d_2 y_2 + \dots + d_k y_k)$  is in nullspace  
 So it  $= c_1 z_1 + c_2 z_2 + \dots + c_m z_m$  same  $c_j$ 's  
 $\Rightarrow y = c_1 z_1 + \dots + c_m z_m + d_1 y_1 + d_2 y_2 + \dots + d_k y_k$  QED

Finish Wednesday notes 6.3.5  
 Fri! finish undetermined coef's  
 (rank + nullify theorem).  
 variation of parameters

do these notes Monday  
 Section 3.6: forced oscillations in mechanical systems (and as we shall see in section 3.7, also in electrical circuits) overview:

We study solutions  $x(t)$  to

$$m x'' + c x' + k x = F_0 \cos(\omega t)$$

using section 3.5 undetermined coefficients algorithms.

- undamped ( $c = 0$ ) :

In this case the complementary homogeneous differential equation for  $x(t)$  is

$$m x'' + k x = 0$$

$$x'' + \frac{k}{m} x = 0$$

$$x'' + \omega_0^2 x = 0$$

which has simple harmonic motion solutions

$$x_H(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) = C_0 \cos(\omega_0 t - \alpha)$$

So for the non-homogeneous DE the section 5.5 method of undetermined coefficients implies we can find particular and general solutions as follows:

- $\omega \neq \omega_0 := \sqrt{\frac{k}{m}} \Rightarrow x_P = A \cos(\omega t)$  because only even derivatives, we don't need

$\sin(\omega t)$  terms !!

$$\Rightarrow x = x_P + x_H = A \cos(\omega t) + C_0 \cos(\omega_0 t - \alpha_0)$$

- $\omega \neq \omega_0$  but  $\omega \approx \omega_0, A \approx C_0$  Beating!
- $\omega = \omega_0$  case 2 section 3.5 undetermined coefficients; since

$$p(r) = r^2 + \omega_0^2 = (r + i\omega_0)^1 (r - i\omega_0)^1$$

our undetermined coefficients guess is

$$x_P = t^1 (A \cos(\omega_0 t) + B \sin(\omega_0 t))$$

$$\Rightarrow x = x_P + x_H = C t \cos(\omega t - \alpha) + C_0 \cos(\omega_0 t - \alpha_0)$$

("pure" resonance!)

- damped ( $c > 0$ ): in all cases  $x_P = A \cos(\omega t) + B \sin(\omega t) = C \cos(\omega t - \alpha)$  (because the roots of the characteristic polynomial are never purely imaginary  $\pm i \omega$  when  $c > 0$ ).

- underdamped:  $x = x_P + x_H = C \cos(\omega t - \alpha) + e^{-p t} C_1 \cos(\omega_1 t - \alpha_1)$
- critically-damped:  $x = x_P + x_H = C \cos(\omega t - \alpha) + e^{-p t} (c_1 t + c_2)$
- over-damped:  $x = x_P + x_H = C \cos(\omega t - \alpha) + c_1 e^{-r_1 t} + c_2 e^{-r_2 t}$