Math 2280-001
Week 6 Feb 13-17, 3.4-3.5, Exam 1 on Friday

Mon Feb 13 Use last Friday's notes to study the unforced mass-spring configuration, section 3.4

Wed Feb 15 Experiments and first midterm review.

Exam 1 is this Friday February 17, from 8:05-9:25 a.m. 8:00 – 9:30, if you want.
I've posted my last two Math 2280 midterms

This exam will cover textbook material from 1.1-1.5, 2.1-2.4, 3.1-3.4. The exam is closed book and closed note. You may use a scientific (but not a graphing) calculator, although symbolic answers are accepted for all problems, so no calculator is really needed.
no 2.5-2.6
(but Euler possible)

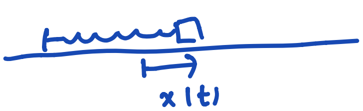
I recommend trying to study by organizing the conceptual and computational framework of the course so far. Only then, test yourself by making sure you can explain the concepts and do typical problems which illustrate them. The class notes and text should have explanations for the concepts, along with worked examples. Old homework assignments and quizzes are also a good source of problems. outline

I will have posted the first exam and solutions, from the last time I taught Math 2280. That exam should give you a feel for how I structure exams and address course topics.

Is there anything from the homework that you'd like to discuss?

3.4.4 : $m = .25$ kg , $F = 9$ N stretches spring .25 m

$F = 9 = kx$
$9 = k(.25)$
$36 = k$



$x(0) = 1$ m
$x'(0) = -5$ m/s.

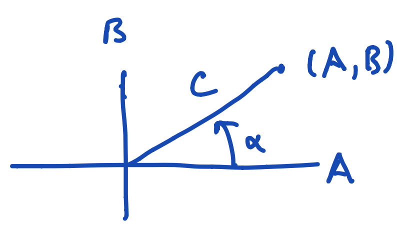
$$mx'' + kx = 0$$
$$x'' + \frac{k}{m}x = 0$$
$$x'' + 144x = 0.$$

$k = 36$
$\frac{k}{m} = \frac{36}{.25} = 144$

$\omega_0 = 12$

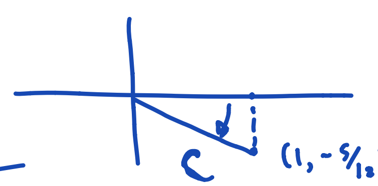
IVP $x(t) = \cos 12t - \frac{5}{12}\sin 12t$
$= C\cos(12t - \alpha)$

$x(t) = A\cos\omega t + B\sin\omega t$
$= C\cos(\omega t - \alpha)$



$A = 1$
$B = -5/12$

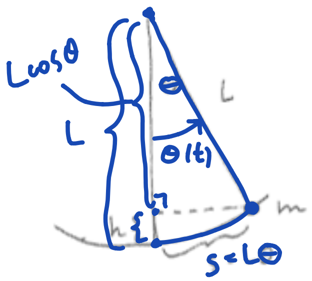
$$\begin{cases} \cos\alpha = \dfrac{1}{C} \\ \sin\alpha = \dfrac{-5/12}{C} \\ \tan\alpha = -5/12 \end{cases}$$



$\alpha = \arctan(-5/12)$

Small oscillation pendulum motion and vertical mass-spring motion are governed by exactly the "same" differential equation that models the motion of the mass in a horizontal mass-spring configuration. The nicest derivation for the pendulum depends on conservation of energy, as indicated below. Conservation of energy is an important tool in deriving differential equations, in a number of different contexts. Today we will test both the pendulum model and the mass-spring model with actual experiments (in the undamped cases), to see if the predicted periods $T = \frac{2\pi}{\omega_0}$ correspond to experimental reality.

1) pendulum



conservative system $KE + PE = \text{const.}$ (total energy is conserved)

$$\underbrace{\tfrac{1}{2} m v^2}_{KE} + \underbrace{mgh}_{PE} = \text{const} = TE$$

$s = L\theta$

$v = \frac{ds}{dt} = L\theta'(t)$

$h = L - L\cos\theta = L(1-\cos\theta)$

$\theta = \theta(t)$

$\frac{d}{dt} s(t) = v$

$\frac{d}{dt} L\theta(t) = L\theta'(t)$

So, $$\underbrace{\tfrac{1}{2} m L^2 (\theta'(t))^2}_{KE} + \underbrace{mgL(1-\cos(\theta(t)))}_{PE} \equiv \text{const}$$

D_t: $m L^2 \theta'\theta'' + mgL(\sin\theta)\theta' \equiv 0$

$$m L\theta'(\underbrace{L\theta'' + g\sin\theta}_{DE}) \equiv 0,$$

$m L\theta' \neq 0$ except at isolated times

$\frac{d}{dt}(\theta'(t))^2 = 2\theta'(t)^1\theta''(t)$

$\frac{d}{dt}(1-\cos\theta(t)) = --(\sin\theta(t))\theta'(t)$

$\rightsquigarrow$ deduce eqtn of motion is

$$\boxed{\theta'' + \frac{g}{L}\sin\theta = 0}$$

$\boxed{\text{linearize}}$

$$\boxed{\theta'' + \frac{g}{L}\theta = 0}$$

$$\boxed{\omega_0 = \sqrt{\frac{g}{L}}}$$

$\theta(t) = C\cos(\omega_0 t - \alpha)$

↘ non-linear DE

but $\sin\theta = \theta - \frac{\theta^3}{3!} + \dots$

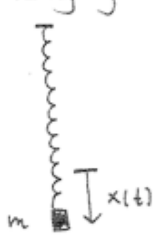
$\sin\theta \approx \theta$ θ small

(is excellent approx (alternating series test)

$|\text{error}| \leq \left|\frac{\theta^3}{3!}\right|$ so if $|\theta| < .1$, $|\text{error}| < \frac{.001}{6}$

(analogous
$x''(t) + \omega_0^2 x = 0$
$x = A\cos\omega_0 t + B\sin\omega_0 t$)

2) hanging mass-spring:



$m x'' = -kx$

$m x'' + kx = 0$

$$\boxed{x'' + \frac{k}{m}x = 0}$$

$\omega_0 = \sqrt{\frac{k}{m}}$.

Why don't you see gravity g in this DE?

Pendulum: measurements and prediction (we'll check these numbers).

```
> restart :
  Digits := 4 :
```

```
> L := 1.526;
  g := 9.806;
```

$\omega := \sqrt{\frac{g}{L}}$; # *radians per second*

$f := evalf(\omega / (2 \cdot \mathrm{Pi}))$; # *cycles per second*

$T := 1/f$; # *seconds per cycle*

$L := 1.526$

$g := 9.806$

$\omega := 2.534945798$ rad/sec

$f := 0.4034491542$ cycles/sec

$T := 2.478627082$ sec/cycle. (1)

$\boxed{2.48 \text{ prediction}}$

10 cycles 25.1
25.02
24.9
~~24.7~~
24.9

$\boxed{\div 10 : 2.49 \text{ or } 2.50 \text{ sec/cycle}}$

experiment

Experiment:

Mass-spring:

compute Hooke's constant:

```
> 98.7 − 83.4; #displacement from extra 50g
```

~~15.3~~ for our spring we measured 15.8 cm (2)

$k := \frac{.05 \cdot 9.806}{.15\cancel{3}8}$; # *solve* $k \cdot x = m \cdot g$ *for k.*

$k := \cancel{3.204575163}$ 3.103 (3)

$m := .1$; # *mass for experiment is 100g*

$\omega := \sqrt{\frac{k}{m}}$; # *predicted angular frequency* $\sqrt{\frac{3.103}{.1}}$

$f := evalf\left(\frac{\omega}{2 \cdot \mathrm{Pi}}\right)$; # *predicted frequency*

$T := \frac{1}{f}$; # *predicted period*

$m := 0.1$

$\omega := 5.660896716$ 5.57 rad/sec

$f := 0.9009596945$ .887 cycles/sec

$T := 1.109927565$ $\boxed{1.128 \text{ sec/cycle}}$ prediction (4)

Experiment: → 20 cycles 22.6
22.9
22.6
22.9
22.7
22.8

$20 \overline{)22.8}$ 1.14

$21 \overline{)2.28}$

$\boxed{1.14 \text{ cycles/sec.}}$

experiment runs a bit slow

We neglected the KE_{spring}, which is small but could be adding intertia to the system and slowing down the oscillations. We can account for this:

<u>Improved mass-spring model</u>

Normalize $TE = KE + PE = 0$ for mass hanging in equilibrium position, at rest. Then for system in motion,

$$KE + PE = KE_{mass} + KE_{spring} + PE_{work}.$$

$$PE_{work} = \int_0^x k\,s\,\mathrm{d}s = \frac{1}{2}k\,x^2, \quad KE_{mass} = \frac{1}{2}m\,(x'(t))^2, \quad KE_{spring} = ????$$

How to model KE_{spring} ? Spring is at rest at top (where it's attached to bar), moving with velocity $x'(t)$ at bottom (where it's attached to mass). Assume it's moving with velocity $\mu\,x'(t)$ at location which is fraction μ of the way from the top to the mass. Then we can compute KE_{spring} as an integral with respect to μ , as the fraction varies $0 \le \mu \le 1$:

$$KE_{spring} = \int_0^1 \frac{1}{2}(\mu\,x'(t))^2 (m_{spring}\,\mathrm{d}\mu)$$

$$= \frac{1}{2}m_{spring}(x'(t))^2 \int_0^1 \mu^2\,d\,\mu = \frac{1}{6}m_{spring}(x'(t))^2 .$$

Thus

$$TE = \frac{1}{2}\left(m + \frac{1}{3}m_{spring}\right)(x'(t))^2 + \frac{1}{2}k\,x^2 = \frac{1}{2}M(x'(t))^2 + \frac{1}{2}k\,x^2 ,$$

where

$$M = m + \frac{1}{3}m_{spring}$$

$D_t(TE) = 0 \Rightarrow$

$$M\,x'(t)\,x''(t) + k\,x(t)\,x'(t) = 0 .$$
$$x'(t)(M\,x'' + k\,x) = 0 .$$

Since $x'(t) = 0$ only at isolated $t-$values, we deduce that the corrected equation of motion is

$$(M\,x'' + k\,x) = 0$$

with

$$\omega_0 = \sqrt{\frac{k}{M}} = \sqrt{\frac{k}{m + \frac{1}{3}m_{spring}}} .$$

Does this lead to a better comparison between model and experiment?

```
> ms := .0103;  # spring has mass 10.3 g
  M := m + 1/3·ms;  # "effective mass"
```

$$ms := 0.0103$$
$$M := 0.1034333333 \quad (5)$$

> $\omega := \sqrt{\frac{k}{M}}$; *# predicted angular frequency*

$\sqrt{3.103}$

$f := evalf\left(\frac{\omega}{2 \cdot \mathrm{Pi}}\right)$; *# predicted frequency*

corrected for our spring

$T := \frac{1}{f}$; *# predicted period*

$$\omega := 5.566150833 \qquad 5.489$$
$$f := 0.8858804190 \qquad .874$$
$$T := 1.128820525 \qquad 1.145 \quad (6)$$

sec/cycle

modified prediction

vs. experiment 1.14

>

Review Questions

1a) What is a differential equation? What is its order? What is an initial value problem, for a first or second order (or higher order) DE?

an equation involving a function $y(x)$ and some of the derivatives of y

the highest order derivative of $y(x)$ appearing in the D.E.

for a 1st order DE, also specify $y(x_0)$ (for 2nd order DE also get to specify $y'(x_0)$)

1b) How do you check whether a function solves a differential equation? An initial value problem?

see if the function makes the DE a true identity

Check if the function has the correct initial value

1c) What is the connection between a first order differential equation and a slope field for that differential equation? The connection between an IVP and the slope field?

- the graph $y=y(x)$ of a solution $y(x)$ to $y'(x)=f(x,y)$ has slope @ (x,y) on the graph equal to the value $f(x,y)$ of the slope function there
- If $y(x)$ solves IVP with $y(x_0)=y_0$, then the graph passes thru (x_0,y_0)

1d) Do you expect solutions to IVP's to exist, at least for values of the input variable close to its initial value? Why? Do you expect uniqueness? What does the existence-uniqueness theorem say? What can cause solutions to not exist beyond a certain input variable value?

We expect solns to the IVP $\begin{cases} y'(x)=f(x,y) \\ y(a)=b \end{cases}$

to exist and be unique. If you can find a coordinate rectangle R, with pt. (a,b) in its interior, so that $f(x,y)$ is continuous in R, then there is a solution to IVP. If $\frac{\partial f}{\partial y}$ is also continuous the solution is unique as long as its graph lies within R

1e) What is Euler's numerical method for approximating solutions to first order IVP's, and how does it relate to slope fields?

$x_{i+1}=x_i+\Delta x$

$y_{i+1}=y_i+f(x_i,y_i)\Delta x$

1f) Can you recognize the first order differential equations for which we've studied solution algorithms, even if the DE is not automatically given to you pre-set up for that algorithm? Do you know the algorithms for solving these particular first order DE's?

- linear §1.5 $y'+P(x)y=Q(x)$
- separable §1.4 $\frac{dy}{dx}=\frac{f(x)}{g(y)}$
- $y'=f(x)$ §1.2

be prepared to recognize whether a DE is linear or separable (and to say what the standard form is, for linear/separable. know how to use the algorithms for finding solutions in each case a number line containing the equilibrium points, and arrows on the intervals between equil points, indicating whether solutions are inc/dec.

2a) *What's an autonomous differential equation? What's an equilibrium solution to an autonomous differential equation? What is a phase diagram for an autonomous first order DE, and how do you construct one? How does a phase diagram help you understand stability questions for equilibria? What does the phase diagram for an autonomous first order DE have to do with the slope field?*

for $y(x)$: $y'(x) = f(y)$

(for $x(t)$: $x'(t) = f(x)$)

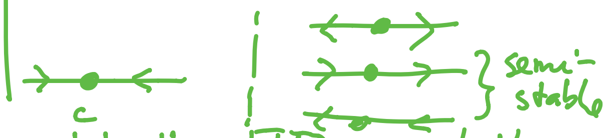
rate of change of solutions only depends on solution value, not on independent variable value.

Note: autonomous $\Rightarrow$ separable

constant solution $y(x) = c$

to find: $y'(x) = f(y)$

for $y(x) = c$ get $0 = f(c)$, so the constant solutions are roots of the slope function



phase diagram is 1-d version that contains essential inc/dec inf from solution graphs on 2-d slope field

2b) *Can you convert a description of a dynamical system in terms of rates of change, or a geometric configuration in terms of slopes, into a differential equation? What are the models we've studied carefully in Chapters 1-2? What sorts of DE's and IVP's arise? Can you solve these basic application DE's, once you've set up the model as a differential equation and/or IVP?*

- input - output - mixing ("tanks")
- improved velocity (add drag)
- improved population
- Newton's law of cooling
- exp growth/decay.

almost always, I've asked (at least) two of these three applications.

see old exams for sample questions

Chapter 3

3a) For functions $y(x)$, why is

$$L(y) := y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y$$

called linear?

$L: V \to W$ is linear means

$$L(y_1 + y_2) = L(y_1) + L(y_2)$$
$$L(cy) = cL(y)$$

3b) For linear operators L, why is the general solution to

$$L(y) = f$$

given by $y = y_p + y_H$, where y_p is any single particular solution, and y_H is the general solution to the homogeneous problem?

$L(y_p) = f$

$L(y_H) = 0$

$\Rightarrow L(y_p + y_H) = L(y_p) + L(y_H) = f + 0$
$= f$

if $L(y_Q) = f$ then $y_Q = y_p + \tilde{y}_H$

$\tilde{y}_H$ is some homog soln.

$y_Q = y_p + (y_Q - y_p)$

$L: \Rightarrow \cancel{L(y_Q) = L(y_p) + L(y_Q - y_p)}$

$\cancel{f = f + L(y_Q - y_p)}$

$\cancel{0 = L(y_Q - y_p)}$

and $L(y_Q - y_p) = f - f$
$= 0$

So $y_Q - y_p = \tilde{y}_H$, a homog. soln.

3c) For the differential operator L above, what is the dimension of the solution space to the homogeneous DE

$$L(y) = 0?$$

n

What does this have to do with the existence-uniqueness theorem?

$\exists!$ theorem says each IVP

$$\begin{cases} y^{(n)} + a_{n-1}y^{(n-1)} + \ldots a_1 y' + a_0 y = 0 \\ y(x_0) = b_0 \\ y'(x_0) = b_1 \\ \vdots \\ y^{(n-1)}(x_0) = b_n \end{cases}$$

has unique solution. So any n solutions $y_1, y_2, \ldots y_n$ such that their initial value vectors at x_0 are a basis for $\mathbb{R}^n$, will be a basis for the solution space

3d) Can you check whether collections of functions are linearly independent?

Wronskian $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ ... etc.

$$\boxed{c_1 y_1 + c_2 y_2 + c_3 y_3 \equiv 0} \overset{?}{\Longrightarrow} c_1 = c_2 = c_3 = 0$$

$\Rightarrow c_1 y_1' + c_2 y_2' + c_3 y_3' \equiv 0$

$\Rightarrow c_1 y_1'' + c_2 y_2'' + c_3 y_3'' \equiv 0$

$$\begin{bmatrix} & W & \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

1. If Wronskian matrix at any x_0 is invertible, deduce $y_1, y_2, \ldots y_n$ are linearly independent

& limiting args

2. If functions have different growth rates as $x \to \infty$ (or $x \to -\infty$) you can show independence

3. By plugging different x-values into the dependency eqtn, you may be able to deduce all $c_j = 0$

(or combine 1, 2, 3)

3e) *What's the Wronskian matrix? How does it arise in studying initial value problems?*

e.g. for $n=2$

$$W(y_1,y_2) = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$$

to solve IVP (e.g. $n=2$) $\begin{cases} y'' + a_1 y' + a_0 y = f \\ y(x_0) = b_0 \\ y'(x_0) = b_1 \end{cases}$

$y = y_P + y_H$

$y = y_P + c_1 y_1 + c_2 y_2$

@ x_0: $\begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} y_P(x_0) \\ y_P'(x_0) \end{bmatrix} + \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

3f) *What's the algorithm for finding the solution space to*

$$L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1\cancel{(x)}y' \cancel{=0} + a_0 y = 0 \quad a_j\text{'s const}$$

(when all the a_j are constants)? What is Euler's formula, and what does it have to do with this discussion? How are repeated roots to the characteristic polynomial handled? Why are the solutions that the algorithm creates linearly independent?

Basis: $y = e^{rx}$ — requires $p(r) = 0$

$r^n + a_{n-1}r^{n-1} + \ldots + a_1 r + a_0 = 0$

$L(y) = e^{rx}(r^n + a_{n-1}r^{n-1} + \ldots + a_1 r + a_0)$ roots

$= e^{rx}(p(r)) \equiv 0$

$r = a \pm bi$

$y_1 = e^{ax}\cos bx$

$y_2 = e^{ax}\sin bx$

$e^{i\theta} = \cos\theta + i\sin\theta$

$\Rightarrow e^{(a\pm bi)x} = e^{ax}e^{\pm ibx} = e^{ax}(\cos bx \pm i\sin bx)$

from which we extract real solutions y_1, y_2

if $p(r)$ has factor $(r-\alpha)^k$ get sol'ns $e^{\alpha x}, xe^{\alpha x}, \ldots, x^{k-1}e^{\alpha x}$

3g) *For the application to unforced (but possibly damped) mass-spring configurations*

$$m\,x''(t) + c\,x'(t) + k\,x(t) = 0$$

what sorts of phenomena arise? Can you convert to amplitude-phase form for simple harmonic motion? Can you describe the important quantities for simple harmonic motion? How are damping phenomena classified? Can you solve IVPs?

$p(r) = mr^2 + cr + k$

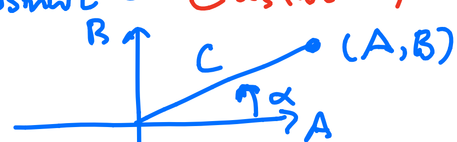
$c > 0$: overdamped $\leftarrow r_1 < r_2 < 0$

crit. damped $\leftarrow$ double root $r_1 < 0$

under damped $\leftarrow$ complex roots $a \pm i\omega_1$ $(a<0)$

$c = 0$: no damping
simple harmonic motion.

$$A\cos\omega t + B\sin\omega t = C\cos(\omega t - \alpha)$$



(from cosine addition angle formula deduce $C = \sqrt{A^2+B^2}$, $\frac{A}{C} = \cos\alpha$, $\frac{B}{C} = \sin\alpha$)