Case 3) $p(r)$ has some complex roots. It turns out that exponential functions e^{rx} still work, except that $r = a \pm b\,i$. However, rather than use those complex exponential functions to construct solution space bases we decompose them into real-valued solutions that are products of exponential and trigonometric functions. We will do the details carefully on Friday - they depend on Euler's (amazing) formula:

$$e^{i\,\theta} = \cos(\theta) + i\sin(\theta).$$

However, the punchline (which you will use in your homework due this Friday), is that if $r = a \pm b\,i$ are two roots of the characteristic polynomial, then

$$y_1(x) = e^{a\,x}\cos(b\,x), y_2(x) = e^{a\,x}\sin(b\,x)$$

both solve the homogeneous differential equation!

More generally, Let L have characteristic polynomial

$$p(r) = r^n + a_{n-1}r^{n-1} + \ldots + a_1 r + a_0$$

with real constant coefficients $a_{n-1},\ldots, a_1, a_0$. If $(r - (a + b\,i))^k$ is a factor of $p(r)$ then so is the conjugate factor $(r - (a - b\,i))^k$. Associated to these two factors are $2\,k$ real and independent solutions to $L(y) = 0$, namely

$$\begin{array}{c} e^{a\,x}\cos(b\,x), e^{a\,x}\sin(b\,x) \\ x\,e^{a\,x}\cos(b\,x), x\,e^{a\,x}\sin(b\,x) \\ \vdots \qquad\qquad \vdots \\ x^{k-1}e^{a\,x}\cos(b\,x), x^{k-1}e^{a\,x}\sin(b\,x) \end{array}$$

Combining cases 1,2,3, yields a complete algorithm for finding the general solution to $L(y) = 0$, as long as you are able to figure out the factorization of the characteristic polynomial $p(r)$.

Exercise 8) Find a basis for the solution space of functions $y(x)$ that solve

$$y'' + 9\,y = 0.$$

$e^{(a \pm bi)x}$ is complex soln

$e^{ax}\cos bx$, $e^{ax}\sin bx$ real solns.

$p(r) = r^2 + 9 = 0$

$r^2 = -9$

$r = \pm 3i$

$a = 0$, $b = 3$

$y_1 = \cos 3x$

$y_2 = \sin 3x$

Exercise 9) Find a basis for the solution space of functions $y(x)$ that solve

$$y'' + 6\,y' + 13\,y = 0.$$

$p(r) = r^2 + 6r + 13 = 0$

$(r+3)^2 + 4 = 0$

$(r+3)^2 = -4$

$r + 3 = \pm 2i$

$r = \underbrace{-3}_{a} \pm \underbrace{2}_{b}i$

$e^{(a+bi)x} = e^{ax}\cos bx + i e^{ax}\sin bx$

$y_1(x) = e^{-3x}\cos 2x$

$y_2(x) = e^{-3x}\sin 2x$

Friday: Today: Wed notes §3.3
§3.3 in Fri notes
Monday: §3.4 Fri notes (below)
Wed: §3.4 experiments
& review.
(old 2280 page)

Math 2280-001
Fri Feb 10
3.3-3.4.

First, Leftovers from section 3.3:

Consider the homogeneous, constant coefficient linear differential equation for $y(x)$,

$$L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_1 y' + a_0 y \equiv 0$$

and its characteristic polynomial

$$L(e^{rx}) = e^{rx}(p(r)); \quad p(r) = r^n + a_{n-1}r^{n-1} + ... + a_1 r + a_0.$$

• Repeated roots mystery: Why, if $(r - r_j)^{k_j}$ is a factor of the characteristic polynomial $p(r)$, do we get k_j linearly independent solutions

$$e^{r_j x}, x e^{r_j x}, x^2 e^{r_j x}, ..., x^{k_j - 1} e^{r_j x} ?$$

to the homogeneous DE? You will explore this mystery in your homework ...

• What to do when the characteristic polynomial has complex roots? We already discussed the fact that if $r = a \pm b\, i$ are complex roots of $p(r)$, then

$r = a \pm i b$

$$y_1(x) = e^{a x}\cos(b x)$$
$$y_2(x) = e^{a x}\sin(b x)$$

are real-valued solutions to the DE. What is behind this mysterious fact? See following ...

(recall, Case 1 was distinct roots to characteristics poly; Case 2 was repeated roots)
Case 3) $p(r)$ has some complex roots. The punch line is that exponential functions $e^{r x}$ still work, except that $r = a \pm b\, i$. However, rather than use those complex exponential functions to construct solution space bases we decompose them into real-valued solutions that are products of exponential and trigonometric functions.

To understand how this all comes about, we need to recall or learn Euler's formula . This also lets us review some important Taylor's series facts from Calc 2. As it turns out, complex number arithmetic and complex exponential functions are very important in many engineering and science applications.

Recall the Taylor-Maclaurin formula from Calculus

$$f(x) \sim f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \frac{1}{3!}f'''(0)x^3 + ... + \frac{1}{n!}f^{(n)}(0)x^n + ...$$

(Recall that the partial sum polynomial through order n matches f and its first n derivatives at $x_0 = 0$. When you studied Taylor series in Calculus you sometimes expanded about points other than $x_0 = 0$. You also need error estimates to figure out on which intervals the Taylor polynomials coverge to f.)

Exercise 1) Use the formula above to recall the three very important Taylor series for

$f(x) = e^x \Rightarrow f^{(n)}(x) = e^x$
$f^{(n)}(0) = 1$

1a) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^k}{k!} + \dots$
1b) $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$
1c) $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

In Calculus you checked that these Taylor series actually converge and equal the given functions, for all real numbers x.

Exercise 2) Let $x = i\,\theta$ and use the Taylor series for e^x as the definition of $e^{i\,\theta}$ in order to derive Euler's formula:

$$\boxed{e^{i\,\theta} = \cos(\theta) + i\sin(\theta)\,.}$$

$(e^{i\pi} = -1)$

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \dots$$
$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{i\theta^7}{7!} + \dots$$

$\cos\theta$

$i\sin\theta$.

From Euler's formula it makes sense to define

$$e^{a + b\,i} := e^a e^{b\,i} = e^a(\cos(b) + i\sin(b))$$

for $a, b \in \mathbb{R}$. So for $x \in \mathbb{R}$ we also get

$r = a \pm bi$:
$$e^{(a \pm b\,i)x} = e^{a\,x}(\cos(b\,x) \pm i\sin(b\,x)) = e^{a\,x}\cos(b\,x) \pm i\,e^{a\,x}\sin(b\,x)\,.$$
$e^{ax}\, e^{\pm ibx}$

For a complex function $f(x) + i\,g(x)$ we define the derivative by

$$D_x(f(x) + i\,g(x)) := f'(x) + i\,g'(x)\,.$$

It's straightforward to verify (but would take some time to check all of them) that the usual differentiation rules, i.e. sum rule, product rule, quotient rule, constant multiple rule, all hold for derivatives of complex functions. The following rule pertains most specifically to our discussion and we should check it:

Exercise 3) Check that $D_x\left(e^{(a + b\,i)x}\right) = (a + b\,i)e^{(a + b\,i)x}$, i.e.

$$\boxed{D_x e^{r\,x} = r\,e^{r\,x}}$$

so $L(e^{rx}) = e^{rx}p(r)$

even if r is complex. (So also $D_x^2\, e^{r\,x} = D_x r\, e^{r\,x} = r^2\, e^{r\,x}$, $D_x^3\, e^{r\,x} = r^3\, e^{r\,x}$, etc.)

$$D_x\left(e^{(a+bi)x}\right) = D_x\left(e^{ax}(\cos bx + i\sin bx)\right)$$
$$= a e^{ax}(\cos bx + i\sin bx) + e^{ax}(-b\sin bx + ib\cos bx)$$
$$\stackrel{?}{=} re^{rx} = (a+bi)(e^{ax})(\cos bx + i\sin bx)$$

Now return to our differential equation questions, with

$$L(y) := y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y\,.$$

Then even for complex $r = a + b\,i$ $(a, b \in \mathbb{R})$, our work above shows that

$$L\left(e^{r x}\right) = e^{r x}\left(r^n + a_{n-1}r^{n-1} + \ldots + a_1 r + a_0\right) = e^{r x}\,p(r)\,.$$

So if $r = a + b\,i$ is a complex root of $p(r)$ then $e^{r x}$ is a complex-valued function solution to $L(y) = 0$. But L is linear, and because of how we take derivatives of complex functions, we can compute in this case that

$$e^{rx} = e^{(a+bi)x} = e^{ax}e^{ibx} = e^{ax}(\cos bx + i\sin bx)$$

$$0 + 0\,i = L\left(e^{r x}\right) = L\left(e^{a x}\cos(b x) + i\,e^{a x}\sin(b x)\right)$$

$$0 + 0i = L\left(e^{a x}\cos(b x)\right) + i\,L\left(e^{a x}\sin(b x)\right)\,.$$

Equating the real and imaginary parts in the first expression to those in the final expression (because that's what it means for complex numbers to be equal) we deduce

$$0 = L\left(e^{a x}\cos(b x)\right)$$
$$0 = L\left(e^{a x}\sin(b x)\right)$$

<u>Upshot:</u> If $r = a + b\,i$ is a complex root of the characteristic polynomial $p(r)$ then

$$y_1 = e^{a x}\cos(b x)$$
$$y_2 = e^{a x}\sin(b x)$$

are two solutions to $L(y) = 0$. (The conjugate root $a - b\,i$ would give rise to y_1, $-y_2$, which have the same span.

$r = a - bi$

$e^{(a-bi)x} = e^{ax}(\cos(-bx) + i\sin(-bx))$
$= e^{ax}\cos bx - i e^{ax}\sin bx$

OR
$z_1 = e^{(a+bi)x}$
$z_1 = e^{ax}\cos bx + i e^{ax}\sin bx$
$z_2 = e^{(a-bi)x}$
$= e^{ax}\cos bx - i e^{ax}\sin bx$
$\frac{z_1+z_2}{2} = e^{ax}\cos bx = y_1$
$\frac{z_1-z_2}{2i} = e^{ax}\sin bx = y_2$

<u>Case 3)</u> Let L have characteristic polynomial

$$p(r) = r^n + a_{n-1}r^{n-1} + \ldots + a_1 r + a_0$$

with real constant coefficients $a_{n-1}, \ldots, a_1, a_0$. If $(r - (a + b\,i))^k$ is a factor of $p(r)$ then so is the conjugate factor $(r - (a - b\,i))^k$. Associated to these two factors are $2\,k$ real and independent solutions to $L(y) = 0$, namely

$$e^{a x}\cos(b x),\ e^{a x}\sin(b x)$$
$$x\,e^{a x}\cos(b x),\ x\,e^{a x}\sin(b x)$$
$$\vdots \qquad\qquad \vdots$$
$$x^{k-1}e^{a x}\cos(b x),\ x^{k-1}e^{a x}\sin(b x)$$

complex sol'ns
$e^{(a\pm bi)x}$
$x e^{(a\pm bi)x}$
$\vdots$

Combining cases 1,2,3, yields a complete algorithm for finding the general solution to $L(y) = 0$, as long as you are able to figure out the factorization of the characteristic polynomial $p(r)$.

<u>Exercise 4)</u> Suppose a 7^{th} order linear homogeneous DE has characteristic polynomial

$$p(r) = \left(r^2 + 6\,r + 13\right)^2 (r - 2)^3\,.$$

What is the general solution to the corresponding homogeneous DE?

$$\left((r+3)^2 + 4\right)^2 (r-2)^3$$
$$(r+3+2i)^2(r+3-2i)^2(r-2)^3$$
$$r = -3 \pm 2i$$

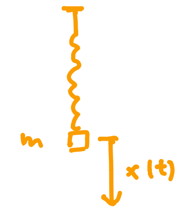
$$y(x) = c_1 e^{-3x}\cos 2x + c_2 e^{-3x}\sin 2x + c_3 e^{2x} + c_4 x e^{2x} + c_5 x^2 e^{2x} + c_6 x e^{-3x}\cos 2x + c_7 x e^{-3x}\sin 2x$$

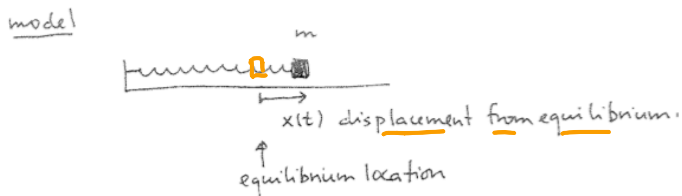
$$\begin{pmatrix} e^{(-3\pm 2i)x} \\ x e^{(-3\pm 2i)x} \end{pmatrix}$$

3.4 applications of constant coefficient homogeneous linear differential equations to unforced mechanical oscillation problems.

In this section we study the differential equation below for functions $x(t)$ measuring the displacement of a mass from its equilibrium solution

$$m\,x'' + c\,x' + k\,x = 0\,.$$





possible external forcing function

Newton's 2^{nd} Law: $mx''(t) = \text{net forces}$

$= -cx' - kx + \cancel{F(t)}$ §3.4.

linear drag, proportional to velocity

force from compression or stretching of spring, relative to equilibrium $x=0$

$x = x(t)$

$mx'' + cx' + kx = 0$

$x'' + \frac{c}{m}x' + \frac{k}{m}x = 0$

In section 3.4 we assume the time dependent external forcing function $F(t) \equiv 0$. The expression for internal forces $-c\,x' - k\,x$ is a linearization model, about the constant solution $x = 0$, $x' = 0$, for which the net forces must be zero (because the configuration stays at rest). Notice that $c \geq 0$, $k > 0$. The actual internal forces are probably not exactly linear, but this model is usually effective when $x(t), x'(t)$ are sufficiently small. k is called the Hooke's constant, and c is called the damping coefficient.

This is a constant coefficient linear homogeneous DE, so we try $x(t) = e^{rt}$ and compute

- $$L(x) := m\,x'' + c\,x' + k\,x = e^{rt}\left(m\,r^2 + c\,r + k\right) = e^{rt}\,p(r).$$

The different behaviors exhibited by solutions to this mass-spring configuration depend on what sorts of roots the characteristic polynomial $p(r)$ processes...

Case 1) no damping ($c = 0$).

$$m\,x'' + k\,x = 0$$

$$x'' + \frac{k}{m}\,x = 0\,.$$

$x = e^{rt}$

$$p(r) = r^2 + \frac{k}{m}\,,$$

has roots

$$r^2 = -\frac{k}{m} \quad \text{i.e.} \quad r = \pm\, i\sqrt{\frac{k}{m}}\,.$$

So the general solution is

$a = 0$

$b = \sqrt{\frac{k}{m}}$

$$x(t) = c_1\cos\left(\sqrt{\frac{k}{m}}\,t\right) + c_2\sin\left(\sqrt{\frac{k}{m}}\,t\right).$$

We write $\sqrt{\frac{k}{m}} := \omega_0$ and call ω_0 the <u>natural angular frequency</u>. Notice that its units are radians per time. We also replace the linear combination coefficients c_1, c_2 by A, B. So, using the alternate letters, the general solution to

$$x'' + \omega_0^{\,2}\, x = 0 \qquad x'' + \frac{k}{m}x = 0$$

is

$$x(t) = \boxed{A\cos(\omega_0 t) + B\sin(\omega_0 t)}$$

This motion is called <u>simple harmonic motion.</u> The reason for this is that $x(t)$ can be rewritten as

$$x(t) = \boxed{C\cos(\omega_0 t - \alpha)} = C\cos(\omega_0 (t - \delta))$$

in terms of an <u>amplitude</u> $C > 0$ and a <u>phase angle</u> α (or in terms of a <u>time delay</u> δ).

To see why functions of the form

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

are equal (for appropriate choices of constants) to ones of the form

$$x(t) = C\cos(\omega_0 t - \alpha)$$

we use the very important the addition angle trigonometry identities, in this case the addition angle for *cosine*: Consider the possible equality of functions

$$\underline{A\cos(\omega_0 t) + B\sin(\omega_0 t)} = \underline{C\cos(\omega_0 t - \alpha)}.$$

<u>Exercise 5)</u> Use the addition angle formula $\cos(a - b) = \cos(a)\cos(b) + \sin(a)\sin(b)$ to show that the two functions above are equal provided

$$A = C\cos\alpha$$
$$B = C\sin\alpha .$$

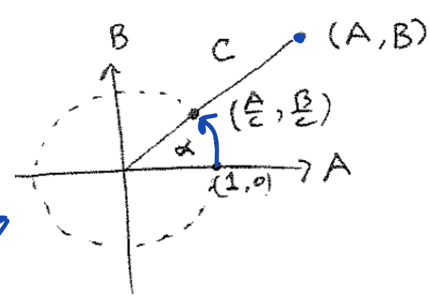
So if C, α are given, the formulas above determine A, B. Conversely, if A, B are given then

$$C = \sqrt{A^2 + B^2}$$
$$\frac{A}{C} = \cos(\alpha),\ \frac{B}{C} = \sin(\alpha)$$

determine C, α. These correspondences are best remembered using a diagram in the $A - B$ plane:

$$\cos(a+b) = \cos a\cos b - \sin a\sin b$$

$$C\cos(w_0 t - \alpha) = C\left(\cos w_0 t\cos(-\alpha) - \sin w_0 t\sin(-\alpha)\right)$$
$$= \boxed{C\cos\alpha}\cos w_0 t + \boxed{C\sin\alpha}\sin w_0 t$$
$$\stackrel{?}{=} \boxed{A}\cos w_0 t + \boxed{B}\sin w_0 t$$



$$A = C\cos\alpha \qquad A^2 + B^2 = C^2$$
$$B = C\sin\alpha \qquad C = \sqrt{A^2+B^2}$$
$$\frac{A}{C} = \cos\alpha$$
$$\frac{B}{C} = \sin\alpha$$

$$e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta} \quad *$$

It is important to understand the behavior of the functions

$$A\cos(\omega_0 t) + B\sin(\omega_0 t) = C\cos(\omega_0 t - \alpha) = C\cos(\omega_0(t - \delta))$$

and the standard terminology:

The <u>amplitude</u> C is the maximum absolute value of $x(t)$. The time delay δ is how much the graph of $C\cos(\omega_0 t)$ is shifted to the right in order to obtain the graph of $x(t)$. Other important data is

ω_0 = ang. freq. $\frac{\text{rad}}{\text{time}}$

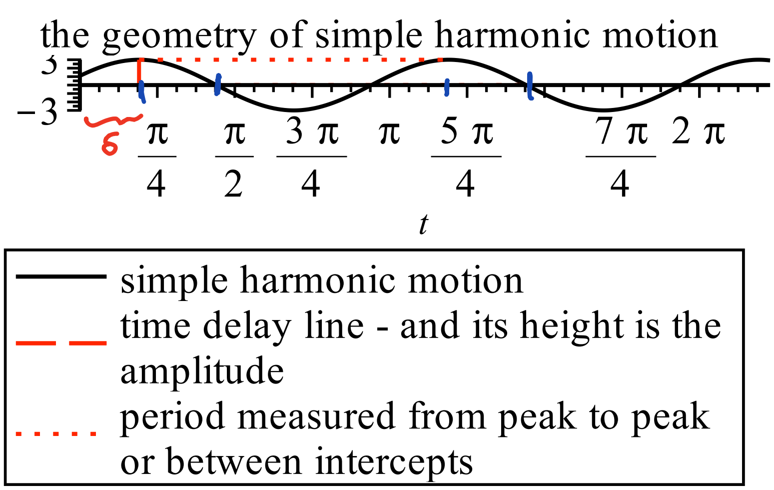
$$f = \text{frequency} = \frac{\omega_0}{2\pi} \quad \text{cycles/time}$$

$$T = \text{period} = \frac{2\pi}{\omega_0} = \text{time/cycle.}$$

$= C\cos(\omega_0(t-\delta))$ $\quad \delta = \frac{\alpha}{\omega_0}$

$C\cos(\omega_0 t - \alpha)$

$C\cos\left(\cancel{\omega_0}\frac{2\pi}{\cancel{\omega_0}} - \alpha\right)$.



(I made that plot above with these commands...and then added a title and a legend, from the plot options.)

```
> with(plots) :
> plot1 := plot(3·cos(2(t − .6)), t = 0 ..7, color = black) :
  plot2 := plot([.6, t, t = 0 ..3.], linestyle = dash) :
  plot3 := plot(3, t = .6 ..(.6) + Pi, linestyle = dot) :
  plot4 := plot(0.02, t = .6 + Pi/4 ...6 + 5·Pi/4, linestyle = dot) :
> display({plot1, plot2, plot3, plot4});
>
```

Exercise 6) A mass of 2 kg oscillates without damping on a spring with Hooke's constant $k = 18\ \frac{N}{m}$. It is initially stretched 1 m from equilibrium, and released with a velocity of $\frac{3}{2}\ \frac{m}{s}$.

$x(0)=1$ $\quad x'(0)=3/2$

6a) Show that the mass' motion is described by $x(t)$ solving the initial value problem

$$\begin{cases} x'' + 9x = 0 \\ x(0) = 1 \\ x'(0) = \frac{3}{2}. \end{cases}$$

6b) Solve the IVP in a, and convert $x(t)$ into amplitude-phase and amplitude-time delay form. Sketch the solution, indicating amplitude, period, and time delay. Check your work with the commands below.

$2x'' + 18x = 0$

units: $mx'' + cx' + kx = 0$

kg $\quad$ m

$N = kg\, m/s^2$

$k = N/m$

$p(r) = r^2 + 9 = 0$

$r = \pm 3i$

$x_H(t) = A\cos 3t + B\sin 3t \quad , \quad x' = -3A\sin 3t + 3B\cos 3t$

$x(0) = 1 = A$

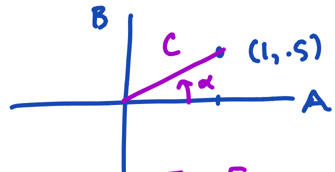
$x'(0) = \frac{3}{2} = 3B$

$A = 1$

$B = \frac{1}{2}.$

$$\boxed{x(t) = \cos 3t + \frac{1}{2}\sin 3t}$$

```
> unassign(`x`);
> with(plots) :
  with(DEtools) :
> dsolve({x''(t) + 9·x(t) = 0, x(0) = 1, x'(0) = 3/2});
> plot(rhs(%), t = 0..5, color = green);
>
```



$C = \sqrt{1 + \frac{1}{4}} = \frac{\sqrt{5}}{2}$

$\alpha = \arctan(.5)$

$\left(\cos\alpha = \frac{1}{\sqrt{5}/2} = \frac{2}{\sqrt{5}} \qquad \sin\alpha = \frac{.5}{\sqrt{5}/2} = \frac{1}{\sqrt{5}} \right)$

• Next, discuss the possibilities that arise when the damping coefficient $c > 0$. There are three cases, depending on the roots of the characteristic polynomial:

Case 2: damping

$$m x'' + c x' + k x = 0$$

$$x'' + \frac{c}{m} x' + \frac{k}{m} x = 0$$

rewrite as

$$x'' + 2 p x' + \omega_0^2 x = 0.$$

$\left(p = \frac{c}{2m}, \omega_0^2 = \frac{k}{m}\right)$. The characteristic polynomial is

$$r^2 + 2 p r + \omega_0^2 = 0$$

which has roots

$mr^2 + cr + k = 0$

$r = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$

$r = -\frac{c}{2m} \pm \sqrt{\frac{c^2}{4m^2} - \frac{k}{m}}$

ω_0^2

$r = -p \pm \sqrt{p^2 - \omega_0^2}$

$p = \frac{c}{2m}$

$$r = -\frac{2p \pm \sqrt{4p^2 - 4\omega_0^2}}{2} = \underline{-p \pm \underbrace{\sqrt{p^2 - \omega_0^2}}}.$$

2a) $(p^2 > \omega_0^2$, or $c^2 > 4\,m\,k)$. $\boxed{\text{overdamped.}}$ In this case we have two negative real roots

$$r_1 < r_2 < 0$$

$-p - \sqrt{p^2 - \omega_0^2} = r_1$

$-p + \sqrt{p^2 - \omega_0^2} = r_2$

and

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} = e^{r_2 t}\left(c_1 e^{(r_1 - r_2)t} + c_2\right).$$

• solution converges to zero exponentially fast; solution passes through equilibrium location $x = 0$ at most once.

2b) $(p^2 = \omega_0^2$, or $c^2 = 4\,m\,k)$ critically damped. Double real root $r_1 = r_2 = -p = -\dfrac{c}{2\,m}$.

$$x(t) = e^{-p\,t}\left(c_1 + c_2\,t\right).$$

• solution converges to zero exponentially fast, passing through $x = 0$ at most once, just like in the overdamped case. The critically damped case is the transition between overdamped and underdamped:

2c) $(p^2 < \omega_0^2$, or $c^2 < 4\,m\,k)$ $\boxed{\text{underdamped}}$ Complex roots

$$r = -p \pm \underbrace{\sqrt{p^2 - \omega_0^2}}_{p^2-\omega_0^2} = -p \pm i\,\omega_1$$

$\omega_1 < \omega_0$

with $\omega_1 = \sqrt{\omega_0^2 - p^2} < \omega_0$.

$$x(t) = e^{-p\,t}\left(A\cos(\omega_1 t) + B\sin(\omega_1 t)\right) = e^{-p\,t}C\cos(\omega_1 t - \alpha_1).$$

• solution decays exponentially to zero, but oscillates infinitely often, with exponentially decaying pseudo-amplitude $e^{-p\,t}C$ and pseudo-angular frequency ω_1, and pseudo-phase angle α_1.

Exercise 7) Classify by finding the roots of the characteristic polynomial. Then solve for $x(t)$:

7a)

$$x'' + 6x' + 9x = 0$$
$$x(0) = 1$$
$$x'(0) = \frac{3}{2}.$$

$p(r) = r^2 + 6r + 9$
$= (r+3)^2$
$x(t) = c_1 e^{-3t} + c_2 t e^{-3t}$

critically damped.

```
> with(DEtools) :
> dsolve({x''(t) + 6·x'(t) + 9·x(t) = 0, x(0) = 1, x'(0) = 3/2});
```

$$\underline{x(t) = e^{-3t} + \frac{9}{2} e^{-3t} t} \tag{2}$$

```
>
```

7b)

$$x'' + 10x' + 9x = 0$$
$$x(0) = 1$$
$$x'(0) = \frac{3}{2}.$$

$p(r) = r^2 + 10r + 9$
$= (r+9)(r+1)$
$x(t) = c_1 e^{-9t} + c_2 e^{-t}$

<u>overdamped</u>

```
> dsolve({x''(t) + 10·x'(t) + 9·x(t) = 0, x(0) = 1, x'(0) = 3/2});
```

$$\underline{x(t) = -\frac{5}{16} e^{-9t} + \frac{21}{16} e^{-t}} \tag{3}$$

```
>
```

7c)

$$x'' + 2x' + 9x = 0$$
$$x(0) = 1$$
$$x'(0) = \frac{3}{2}.$$

$p(r) = r^2 + 2r + 9$
$= (r+1)^2 + 8$
$= (r+1+2\sqrt{2}i)(r+1-2\sqrt{2}i)$
$r = \underbrace{-1}\pm\underbrace{2\sqrt{2}}i$

underdamped.

```
> dsolve({x''(t) + 2·x'(t) + 9·x(t) = 0, x(0) = 1, x'(0) = 3/2});
```

$$x(t) = \frac{5}{8}\sqrt{2}\,\underbrace{e^{-t}}\sin\left(2\sqrt{2}\,t\right) + \underbrace{e^{-t}}\cos\left(2\sqrt{2}\,t\right) \tag{4}$$

```
>
> with(plots) :
> plot0 := plot(cos(3·t) + 1/2·sin(3·t), t = 0..4, color = red) :
```

$plot1a := plot\left(\exp(-3\cdot t)\cdot\left(1+\frac{9}{2}\cdot t\right), t=0..4, color=green\right):$

$plot1b := plot\left(\frac{21}{16}\cdot\exp(-t)-\frac{5}{16}\cdot\exp(-9\cdot t), t=0..4, color=blue\right):$

$plot1c := plot\left(\frac{5}{8}\cdot\sqrt{2}\,e^{-t}\cdot\sin\left(2\sqrt{2}\cdot t\right)+e^{-t}\cdot\cos\left(2\sqrt{2}\cdot t\right), t=0..4, color=black\right):$

$display(\{plot0, plot1a, plot1b, plot1c\}, title = `IVP with all damping possibilities`);$