Method 2) If your interval stretches to $+\infty$ or to $-\infty$ and your functions grow at different rates, you may be able to take limits (after dividing the dependency equation by appropriate functions of x), to deduce independence.

Exercise 2) Use method 2 for $I = \mathbb{R}$, to show that the functions

$$y_1(x) = 1,\ y_2(x) = x,\ y_3(x) = x^2$$

are linearly independent. Hint: first divide the dependency equation by the fastest growing function, then let $x \to \infty$.

$$c_1 \cdot 1 + c_2 x + c_3 x^2 \equiv 0$$

$\div x^2$ $(x \neq 0)$ $\quad \dfrac{c_1}{x^2} + \dfrac{c_2}{x} + c_3 \equiv 0$

$\lim_{x\to\infty}$: $\Rightarrow c_3 = 0$, so $\quad c_1 + c_2 x \equiv 0$

$\div x \quad \dfrac{c_1}{x} + c_2 \equiv 0$

$\lim_{x\to\infty}$: $\Rightarrow c_2 = 0$

$\Rightarrow c_1 = 0$

$$c_1 e^{-3x} + c_2 e^{x} + c_3 e^{-x} \equiv 0$$

$\div e^x$: $\quad c_1 e^{-4x} + c_2 + c_3 e^{-2x} \equiv 0$

$\lim_{x\to\infty}$ $\Rightarrow c_2 = 0$

Method 3) If

$$c_1 y_1 + c_2 y_2 + \ldots + c_n y_n = 0$$

$\forall\, x \in I$, then we can take derivatives to get a system

$$\begin{aligned}
c_1 y_1 \; + \; c_2 y_2 + \ldots + \; c_n y_n &= 0\\
c_1 y_1{}' + c_2 y_2{}' + \ldots + c_n y_n{}' &= 0\\
c_1 y_1{}'' + c_2 y_2{}'' + \ldots + c_n y_n{}'' &= 0\\
&\\
c_1 y_1^{(n-1)} + c_2 y_2^{(n-1)} + \ldots + c_n y_n^{(n-1)} &= 0
\end{aligned}$$

(We could keep going, but stopping here gives us n equations in n unknowns.)
Plugging in any value of x_0 yields a homogeneous algebraic linear system of n equations in n unknowns, which is equivalent to the Wronskian matrix equation

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) & \ldots & y_n(x_0) \\ y_1{}'(x_0) & y_2{}'(x_0) & \ldots & y'_n(x_0) \\ \vdots & \vdots & \ldots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \ldots & y_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

If this Wronskian matrix is invertible at even a single point $x_0 \in I$, then the functions are linearly independent! (So if the determinant is zero at even a single point $x_0 \in I$, then the functions are independent....strangely, even if the determinant was zero for all $x \in I$, then it could still be true that the functions are independent....but that won't happen if our n functions are all solutions to the same n^{th} order linear homogeneous DE.)

Exercise 3) Use method 3 for $I = \mathbb{R}$, to show that the functions

$$y_1(x) = 1,\ y_2(x) = x,\ y_3(x) = x^2$$

are linearly independent. Use $x_0 = 0$.

$$W = \begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix} = \begin{bmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{bmatrix}$$

@ $x_0 = 0$, $W = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is non-singular ($\det W = 2$)

Remark 1) Method 3 is usually not the easiest way to prove independence in general. But we and the text like it when studying differential equations because as we've seen, the Wronskian matrix shows up when you're trying to solve initial value problems using

$$y = y_P + y_H = y_P + c_1 y_1 + c_2 y_2 + \ldots + c_n y_n \qquad y$$

as the general solution to

$$y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = f.$$

This is because, if the initial conditions for this inhomogeneous DE are

$$y(x_0) = b_0, y'(x_0) = b_1, \ldots y^{(n-1)}(x_0) = b_{n-1} \qquad \text{IVP}$$

then you need to solve matrix algebra problem

$$\begin{array}{c} y \\ y' \\ \\ \\ \end{array} \rightarrow \begin{bmatrix} y_P(x_0) \\ y_P'(x_0) \\ \vdots \\ y_P^{(n-1)}(x_0) \end{bmatrix} + \begin{bmatrix} y_1(x_0) & y_2(x_0) & \cdots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \cdots & y_n'(x_0) \\ \vdots & \vdots & \cdots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \cdots & y_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix}. \quad \begin{array}{c} y(x_0) \\ y'(x_0) \\ \\ y^{(n-1)}(x_0) \end{array}$$

for the vector $[c_1, c_2, \ldots, c_n]^T$ of linear combination coefficients. And so if you're using the Wronskian matrix method, and the matrix is invertible at x_0 then you are effectively directly checking that $y_1, y_2, \ldots y_n$ are a basis for the homogeneous solution space, and because you've found the Wronskian matrix you are ready to solve any initial value problem you want by solving for the linear combination coefficients above.

Remark 2) There is a seemingly magic consequence in the situation above, in which $y_1, y_2, \ldots y_n$ are all solutions to the same n^{th}-order homogeneous DE

$$y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0$$

(even if the coefficients aren't constants): If the Wronskian matrix of your solutions $y_1, y_2, \ldots y_n$ is invertible at a single point x_0, then $y_1, y_2, \ldots y_n$ are a basis because linear combinations uniquely solve all IVP's at x_0. But since they're a basis, that also means that linear combinations of $y_1, y_2, \ldots y_n$ solve all IVP's at any other point x_1. This is only possible if the Wronskian matrix at x_1 also reduces to the identity matrix at x_1 and so is invertible there too. In other words, the Wronskian determinant will either be non-zero $\forall\, x \in I$, or zero $\forall\, x \in I$, when your functions $y_1, y_2, \ldots y_n$ all happen to be solutions to the same n^{th} order homogeneous linear DE as above.

Exercise 4) Verify that $y_1(x) = 1,\ y_2(x) = x,\ y_3(x) = x^2$ all solve the third order linear homogeneous DE

$$y''' = 0,$$

and that their Wronskian determinant is indeed non-zero $\forall\, x \in \mathbb{R}$.

$$W(y_1, y_2, y_3) = \begin{bmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{bmatrix}.$$

is non-singular $\forall x$!

3.3: Algorithms for a basis and the general (homogeneous) solution to

$$L(y) := y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0$$

when the coefficients $a_{n-1}, a_{n-2}, \ldots a_1, a_0$ are all constant.

strategy: Try to find a basis made of exponential functions....try $y(x) = e^{rx}$. In this case

$L(e^{rx})$ $\quad L(y) = e^{rx}\left(r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0\right) = e^{rx} p(r)$.

We call this polynomial $p(r)$ the characteristic polynomial for the differential equation, and can read off what it is directly from the expression for $L(y)$. For each root r_j of $p(r)$, we get a solution $e^{r_j x}$ to the homogeneous DE.

Case 1) If $p(r)$ has n distinct (i.e. different) real roots $r_1, r_2, \ldots, r_n$, then

$$e^{r_1 x}, e^{r_2 x}, \ldots, e^{r_n x}$$

$p(r) = (r-r_1)(r-r_2) \cdots (r-r_n)$

$r_1 < r_2 < \ldots < r_n$.

is a basis for the solution space; i.e. the general solution is given by

$$y_H(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \ldots + c_n e^{r_n x}.$$

Exercise 5) By construction, $e^{r_1 x}, e^{r_2 x}, \ldots, e^{r_n x}$ all solve the differential equation. Show that they're linearly independent. This will be enough to verify that they're a basis for the solution space, since we know the solution space is n-dimensional. Hint: The easiest way to show this is to list your roots so that $r_1 < r_2 < \ldots < r_n$ and to use a limiting argument, as we did in Method 2 and Exercise 2 above.

$$c_1 e^{r_1 x} + c_2 e^{r_2 x} + \cdots + c_n e^{r_n x} \equiv 0 \quad (= \forall x)$$

$L(y) = 0$

domain was $x \in \mathbb{R}$.

$$\div e^{r_n x}: \quad c_1 e^{(r_1 - r_n)x} + c_2 e^{(r_2 - r_n)x} + \cdots + c_n \equiv 0$$

each $r_j - r_n < 0$ since $r_j < r_n$ $(j < n)$.

$$\lim_{x \to \infty} \left(0 \qquad\qquad \right) = 0$$

$$0 + 0 + \cdots + 0 + c_n = 0 \Rightarrow c_n = 0$$

now repeat arg, except $\div e^{r_{n-1} x} \Rightarrow c_{n-1} = 0$

$\vdots$

$c_1 = 0$.

Case 2) Repeated real roots. In this case $p(r)$ has all real roots $r_1, r_2, \ldots r_m$ $(m < n)$ with the r_j all different, but some of the factors $(r - r_j)$ in $p(r)$ appear with powers bigger than 1. In other words, $p(r)$ factors as

$$p(r) = (r - r_1)^{k_1}(r - r_2)^{k_2}\cdots(r - r_m)^{k_m}$$

with some of the $k_j > 1$, and $k_1 + k_2 + \ldots + k_m = n$.

, solns $e^{r_1 x}, e^{r_2 x}, \ldots e^{r_m x}$
$m<n$ not enough!

Start with a small example: The case of a second order DE for which the characteristic polynomial has a double root. There is an example like this in your section 3.1 homework for Friday.

Exercise 6) Let r_1 be any real number. Consider the homogeneous DE

$$L(y) := y'' - 2r_1 y' + r_1^2 y = 0.$$

with $p(r) = r^2 - 2r_1 r + r_1^2 = (r - r_1)^2$, i.e. r_1 is a double root for $p(r)$. Show that $e^{r_1 x}, x e^{r_1 x}$ are a basis for the solution space to $L(y) = 0$, so the general homogeneous solution is $y_H(x) = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$. Start by checking that $x e^{r_1 x}$ actually (magically?) solves the DE.

$L(y) = y'' - y' - 2y = 0$

$y = e^{rx}$

$p(r) = r^2 - r - 2 = 0$

$\quad = (r-2)(r+1) = 0.$

$y_1 = e^{2x}, \quad y_2 = e^{-x}$

$y_H(x) = c_1 e^{2x} + c_2 e^{-x}$

Why? operator ideas

$D = \frac{d}{dx}, \quad Dy := y'$

$I, \quad Iy := y$

$L(y) = y'' - y' - 2y$

$\quad = D^2 y - Dy - 2Iy$

$\quad = [D^2 - D - 2I]y$

$L = D^2 - D - 2I$

$L = (D - 2I)\circ(D + I) = (D+I)\circ(D-2I)$

(check: $(D-2I)\circ(D+I)(y)$

$= (D-2I)(y'+y)$

$= D(y'+y) - 2I(y'+y)$

$= y'' + y' - 2y' - 2y$

$= y'' - y' - 2y$ ✓)

$(D - aI)e^{ax} = ae^{ax} - ae^{ax} = 0$

∴ $L(e^{2x}) = 0, \; L(e^{-x}) = 0$

($L(y) = y'' - 6y' + 9y = 0$

$y = e^{rx}$

$p(r) = r^2 - 6r + 9$

$\quad = (r-3)^2$

$y_1 = e^{3x}$

$9\,(\, y_2 = x e^{3x}\,)$ "magic"

$-6\,(\, y_2' = e^{3x}(1+3x)\,)$

~~$y_2'' = e^{3x}[3(1+3x)+3]$~~

$+1\,(\, y_2'' = e^{3x}[6 + 9x]\,)$

$y_2'' - 6y_2' + 9y_2 = e^{3x}x\,(\cancel{9 - 18} + 9) \to 0$

$\quad + e^{3x}(\cancel{-6+6}) \to 0$

$L = D^2 - 6D + 9I$

$L = (D - 3I)\circ(D - 3I)$

$(D-3I)e^{3x} = 0$

$(D-3I)\,xe^{3x} = (1e^{3x} + \cancel{3xe^{3x}} - \cancel{3xe^{3x}})$

$L(xe^{3x}) = (D-3I)\circ\underbrace{(D-3I)(xe^{3x})}_{e^{3x}}$

$= (D-3I)e^{3x} = 0$

$(D - aI)(f(x)e^{ax}) = f'(x)e^{ax} + \cancel{af(x)e^{ax}} - \cancel{af(x)e^{ax}}$

e.g. $(D - aI)^3 x^2 e^{ax} = (D-aI)^2 2x e^{ax}$

$= (D-aI)2e^{ax} = 0$

Here's the general algorithm for repeated real roots: If the characteristic polynomial

$$p(r) = (r - r_1)^{k_1}(r - r_2)^{k_2}\cdots(r - r_m)^{k_m},$$

then (as before) $e^{r_1 x}, e^{r_2 x}, \ldots, e^{r_m x}$ are independent solutions, but since $m < n$ there aren't enough of them to be a basis for the n-dimensional solution space. Here's how you get the rest: For each $k_j > 1$, you actually get independent solutions

$$e^{r_j x}, x e^{r_j x}, x^2 e^{r_j x}, \ldots, x^{k_j - 1} e^{r_j x}.$$

This yields k_j solutions for each root r_j, so since $k_1 + k_2 + \ldots + k_m = n$ you get a total of n solutions to the differential equation. There's a good explanation as to why these additional functions actually do solve the differential equation, see page 165 and the discussion of "polynomial differential operators". I'll make a homework problem in your next assignment to explore these ideas, and discuss it a bit in class, on Friday. Using the limiting method we discussed earlier, it's not too hard to show that all n of these solutions are indeed linearly independent, so they are in fact a basis for the solution space to $L(y) = 0$.

<u>Exercise 7)</u> Explicitly antidifferentiate to show that the solution space to the differential equation for $y(x)$

$$L(y) \quad y^{(4)} - y^{(3)} = 0$$

agrees with what you would get using the repeated roots algorithm in <u>Case 2</u> above. Hint: first find $v = y'''$, using $v' - v = 0$, then antidifferentiate three times to find y_H. When you compare to the repeated roots algorithm, note that it includes the possibility $r = 0$ and that $e^{0x} = 1$, $xe^{0x} = x$, etc.

<u>recipe</u>: $y = e^{rx}$

$L(y) = e^{rx}\left(\underbrace{r^4 - r^3}\right) \equiv 0$

$p(r) = r^3(r-1)$

$= (r-0)^3(r-1)$

roots $r = 0, 1.$

$y_H(x) = c_1 e^x + c_2 \underbrace{e^{0x}}_{1} + c_3 x + c_4 x^2$

basis: $\{e^x, 1, x, x^2\}$

chapter 1 way.

Let $v = y'''$, for soln y

then $v' - v = 0$

$v'(x) = v(x)$

$v(x) = Ce^x$

$y''' = Ce^x$

$y'' = Ce^x + c_1$

$y' = Ce^x + c_1 x + c_2$

$y = Ce^x + \frac{c_1}{2}x^2 + c_2 x + c_3$

Case 3) $p(r)$ has some complex roots. It turns out that exponential functions $e^{r x}$ still work, except that $r = a \pm b\,i$. However, rather than use those complex exponential functions to construct solution space bases we decompose them into real-valued solutions that are products of exponential and trigonometric functions. We will do the details carefully on Friday - they depend on Euler's (amazing) formula:

$$e^{i\,\theta} = \cos(\theta) + i \sin(\theta)\ .$$

However, the punchline (which you will use in your homework due this Friday), is that if $r = a \pm b\,i$ are two roots of the characteristic polynomial, then

$$y_1(x) = e^{a x}\cos(b\,x),\ y_2(x) = e^{a x}\sin(b\,x)$$

both solve the homogeneous differential equation!

More generally, Let L have characteristic polynomial

$$p(r) = r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0$$

with real constant coefficients $a_{n-1}, \ldots, a_1, a_0$. If $(r - (a + b\,i))^k$ is a factor of $p(r)$ then so is the conjugate factor $(r - (a - b\,i))^k$. Associated to these two factors are $2\,k$ real and independent solutions to $L(y) = 0$, namely

$$\begin{array}{c} e^{a x}\cos(b\,x),\ e^{a x}\sin(b\,x) \\ x\,e^{a x}\cos(b\,x),\ x\,e^{a x}\sin(b\,x) \\ \vdots \qquad\qquad \vdots \\ x^{k-1} e^{a x}\cos(b\,x),\ x^{k-1} e^{a x}\sin(b\,x) \end{array}$$

Combining cases 1,2,3, yields a complete algorithm for finding the general solution to $L(y) = 0$, as long as you are able to figure out the factorization of the characteristic polynomial $p(r)$.

Exercise 8) Find a basis for the solution space of functions $y(x)$ that solve

$$y'' + 9\,y = 0\ .$$

$e^{(a \pm bi)x}$ is complex soln

$e^{ax}\cos bx$
$e^{ax}\sin bx$ real solns.

$p(r) = r^2 + 9 = 0$
$r^2 = -9$
$r = \pm 3i$

$a = 0$
$b = 3$

$y_1 = \cos 3x$
$y_2 = \sin 3x$

Exercise 9) Find a basis for the solution space of functions $y(x)$ that solve

$$y'' + 6\,y' + 13\,y = 0\ .$$

Exercise 10) Suppose a 7^{th} order linear homogeneous DE has characteristic polynomial

$$p(r) = \left(r^2 + 6\,r + 13\right)^2 (r-2)^3 .$$

What is the general solution to the corresponding homogeneous DE?

Friday: Today: Wed notes §3.3
§3.3 in Fri notes
Monday: §3.4 Fri notes (below)
Wed: §3.4 experiments
& review.
(old 2280 page)

Math 2280-001
Fri Feb 10
3.3-3.4.

First, Leftovers from section 3.3:

Consider the homogeneous, constant coefficient linear differential equation for $y(x)$,

$$y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_1 y' + a_0 y \equiv 0$$

and its characteristic polynomial

$$p(r) = r^n + a_{n-1} r^{n-1} + ... + a_1 r + a_0 .$$

• Repeated roots mystery: Why, if $\left(r - r_j\right)^{k_j}$ is a factor of the characteristic polynomial $p(r)$, do we get k_j linearly independent solutions

$$e^{r_j x}, x e^{r_j x}, x^2 e^{r_j x}, ..., x^{k_j - 1} e^{r_j x} ?$$

to the homogeneous DE? You will explore this mystery in your homework ...

• What to do when the characteristic polynomial has complex roots? We already discussed the fact that if $r = a \pm b\, i$ are complex roots of $p(r)$, then

$$y_1(x) = e^{a x} \cos(b x)$$
$$y_2(x) = e^{a x} \sin(b x)$$

are real-valued solutions to the DE. What is behind this mysterious fact? See following ...

(recall, Case 1 was distinct roots to characteristics poly; Case 2 was repeated roots)
Case 3) $p(r)$ has some complex roots. The punch line is that exponential functions $e^{r x}$ still work, except that $r = a \pm b\, i$. However, rather than use those complex exponential functions to construct solution space bases we decompose them into real-valued solutions that are products of exponential and trigonometric functions.

To understand how this all comes about, we need to recall or learn Euler's formula . This also lets us review some important Taylor's series facts from Calc 2. As it turns out, complex number arithmetic and complex exponential functions are very important in many engineering and science applications.

Recall the Taylor-Maclaurin formula from Calculus

$$f(x) \sim f(0) + f'(0)x + \frac{1}{2!} f''(0)x^2 + \frac{1}{3!} f'''(0)x^3 + ... + \frac{1}{n!} f^{(n)}(0)x^n + ...$$

(Recall that the partial sum polynomial through order n matches f and its first n derivatives at $x_0 = 0$. When you studied Taylor series in Calculus you sometimes expanded about points other than $x_0 = 0$. You also need error estimates to figure out on which intervals the Taylor polynomials coverge to f.)

Exercise 1) Use the formula above to recall the three very important Taylor series for

$f(x) = e^x \Rightarrow f^{(n)}(x) = e^x$
$f^{(n)}(0) = 1$

1a) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^k}{k!} + \cdots$
1b) $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$
1c) $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

In Calculus you checked that these Taylor series actually converge and equal the given functions, for all real numbers x.

Exercise 2) Let $x = i\,\theta$ and use the Taylor series for e^x as the definition of $e^{i\,\theta}$ in order to derive Euler's formula:

$$\boxed{e^{i\,\theta} = \cos(\theta) + i\sin(\theta)\,.} \qquad \left(e^{i\pi} = -1\right)$$

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots$$
$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{i\theta^7}{7!} + \cdots$$

$\cos\theta$

$i\sin\theta$.

From Euler's formula it makes sense to define

$$e^{a + b\,i} := e^a e^{b\,i} = e^a(\cos(b) + i\sin(b))$$

for $a, b \in \mathbb{R}$. So for $x \in \mathbb{R}$ we also get

$$e^{(a + b\,i)x} = e^{a\,x}(\cos(b\,x) + i\sin(b\,x)) = e^{a\,x}\cos(b\,x) + i\,e^{a\,x}\sin(b\,x)\,.$$

For a complex function $f(x) + i\,g(x)$ we define the derivative by

$$D_x(f(x) + i\,g(x)) := f'(x) + i\,g'(x)\,.$$

It's straightforward to verify (but would take some time to check all of them) that the usual differentiation rules, i.e. sum rule, product rule, quotient rule, constant multiple rule, all hold for derivatives of complex functions. The following rule pertains most specifically to our discussion and we should check it:

Exercise 3) Check that $D_x\left(e^{(a + b\,i)x}\right) = (a + b\,i)e^{(a + b\,i)x}$, i.e.

$$D_x\, e^{r\,x} = r\,e^{r\,x}$$

even if r is complex. (So also $D_x^2\, e^{r\,x} = D_x r\, e^{r\,x} = r^2\, e^{r\,x}$, $D_x^3\, e^{r\,x} = r^3\, e^{r\,x}$, etc.)

Now return to our differential equation questions, with
$$L(y) := y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y .$$
Then even for complex $r = a + b\,i \quad (a, b \in \mathbb{R})$, our work above shows that
$$L\left(e^{r x}\right) = e^{r x}\left(r^n + a_{n-1}r^{n-1} + \ldots + a_1 r + a_0\right) = e^{r x}\,p(r) .$$
So if $r = a + b\,i$ is a complex root of $p(r)$ then $e^{r x}$ is a complex-valued function solution to $L(y) = 0$.
But L is linear, and because of how we take derivatives of complex functions, we can compute in this case that

$e^{rx} = e^{(a+bi)x} = e^{ax}e^{ibx} = e^{ax}(\cos bx + i\sin bx)$

$$0 + 0\,i = L\left(e^{r x}\right) = L\left(e^{a x}\cos(b x) + i\,e^{a x}\sin(b x)\right)$$
$$= L\left(e^{a x}\cos(b x)\right) + i\,L\left(e^{a x}\sin(b x)\right) .$$
Equating the real and imaginary parts in the first expression to those in the final expression (because that's what it means for complex numbers to be equal) we deduce
$$0 = L\left(e^{a x}\cos(b x)\right)$$
$$0 = L\left(e^{a x}\sin(b x)\right) .$$
<u>Upshot:</u> If $r = a + b\,i$ is a complex root of the characteristic polynomial $p(r)$ then
$$y_1 = e^{a x}\cos(b x)$$
$$y_2 = e^{a x}\sin(b x)$$
are two solutions to $L(y) = 0$. (The conjugate root $a - b\,i$ would give rise to $y_1, -y_2$, which have the same span.

<u>Case 3)</u> Let L have characteristic polynomial
$$p(r) = r^n + a_{n-1}r^{n-1} + \ldots + a_1 r + a_0$$
with real constant coefficients $a_{n-1}, \ldots, a_1, a_0$. If $(r - (a + b\,i))^k$ is a factor of $p(r)$ then so is the conjugate factor $(r - (a - b\,i))^k$. Associated to these two factors are $2\,k$ real and independent solutions to $L(y) = 0$, namely
$$e^{a x}\cos(b x),\ e^{a x}\sin(b x)$$
$$x\,e^{a x}\cos(b x),\ x\,e^{a x}\sin(b x)$$
$$\vdots \qquad\qquad \vdots$$
$$x^{k-1}e^{a x}\cos(b x),\ x^{k-1}e^{a x}\sin(b x)$$

Combining cases 1,2,3, yields a complete algorithm for finding the general solution to $L(y) = 0$, as long as you are able to figure out the factorization of the characteristic polynomial $p(r)$.

<u>Exercise 4)</u> Suppose a 7^{th} order linear homogeneous DE has characteristic polynomial
$$p(r) = \left(r^2 + 6\,r + 13\right)^2 (r - 2)^3 .$$
What is the general solution to the corresponding homogeneous DE?