Math 2280-001
Optional Extra Credit homework assignment, due Monday May 1 at 6:00 p.m.
(20 points possible; will replace lowest remaining hw score)

1) Partial fractions (6 points)
This problem explores why the algorithm for partial fractions works, and is based on linear algebra ideas.

The easiest sort of partial fractions problem is when the denominator has distinct roots. For example, if the the denominator is a cubic, this would be the situation of

$$\frac{p_2(s)}{(s-\alpha)\cdot(s-\beta)\cdot(s-\gamma)} = \frac{A}{s-\alpha} + \frac{B}{s-\beta} + \frac{C}{s-\gamma}$$

with $p_2(s) = a_0 + a_1 s + a_2 s^2$ a polynomial of degree at most 2, and α, β, γ distinct roots. Equating numerators after cross multiplication, one is led to finding A, B, C so that

$$a_0 + a_1 s + a_2 s^2 = A(s-\beta)(s-\gamma) + B(s-\alpha)(s-\gamma) + C(s-\alpha)(s-\beta)$$

holds for all s. On the left of this identity we have $p_2(s)$ expressed as a linear combination of the "standard" polynomial basis functions $\{1, s, s^2\}$ for the space P_2 of polynomials of degree less than or equal to two. On the right, we are trying to express $p_2(s)$ as a linear combination of three other polynomials that are potentially an alternate basis for P_2:

$$\{q_1(s), q_2(s), q_3(s)\} = \{(s-\beta)(s-\gamma),\ (s-\alpha)(s-\gamma),\ (s-\alpha)(s-\beta)\}$$

1a) Show that $\{q_1(s), q_2(s), q_3(s)\}$ is linearly independent. This implies this collection automatically also span the three-dimensional space P_2 as well, so are a basis, so also that the partial fractions problem always has a unique solution (which you can then find quickly in this case by substituting in appropriate values for s or by the other method you've learned.) Hint: Let a linear combination be the zero function, and show all the coefficients must be zero, using substitutions:

$$c_1 q_1(s) + c_2 q_2(s) + c_3 q_3(s) \equiv 0 \Rightarrow c_1 = c_2 = c_3 = 0 .$$

1b) Use substitutions to find the partial fraction decomposition of

$$\frac{s+3}{s(s^2+1)}$$

by factoring the denominator as $s(s+i)(s-i)$. Then rewrite your answer in the usual real form for partial fractions.

1c) The situation with repeated roots and the partial fractions algorithm is similar. For simplicity we'll consider real roots. For example, show that the corresponding numerator functions below constitute a basis for the five-dimensional space P_4 of polynomials of degree at most 4, by showing they're linearly independent: In other words, if we wanted to do partial fractions for

$$\frac{p_4(s)}{(s-4)^2(s-3)^3} = \frac{A}{s-4} + \frac{B}{(s-4)^2} + \frac{C}{s-3} + \frac{D}{(s-3)^2} + \frac{E}{(s-3)^3}$$

we'd want the functions below to be a basis for P_4. So, we'd need to know they were linearly indpendent. So, use substitution and successive differentiation and substitution to show that A,B,C,D,E must be zero if the identity below holds:

$$0 \equiv A(s-4)\cdot(s-3)^3 + B(s-3)^3 + C(s-4)^2(s-3)^2 + D(s-4)^2(s-3) + E(s-4)^2.$$

(Finding the unique values of A, B, C, D, E for a specific partial fractions problem could involve somewhat more work.)

2) (14 points) This problem goes through some of the basics for inner product spaces.

Definition Let V is any real-scalar vector space. we call V an inner product space if there is an inner product $\langle f, g\rangle$
for which the inner product satisfies $\forall\, f, g, h \in V$ and scalars $s \in \mathbb{R}$:
a) $\langle f, f\rangle \geq 0$. $\langle f, f\rangle = 0$ if and only if $f = 0$.
b) $\langle f, g\rangle = \langle g, f\rangle$.
c) $\langle f, g + h\rangle = \langle f, g\rangle + \langle f, h\rangle$
d) $\langle (s f), g\rangle = s\langle f, g\rangle = \langle f, s\, g\rangle$.

In this case one can define $\| f \| = \sqrt{\langle f, f\rangle}$, $dist(f, g) = \| f - g \|$; prove the Cauchy-Schwarz inequality and the triangle inequalities; define angles between vectors, and in particular, orthogonality between vectors; find ortho-normal bases $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots \boldsymbol{u}_n\}$ for finite-dimensional subspaces W, and prove that for any $f \in V$ the nearest element in W to f is given by

$$proj_W f = \langle f, u_1\rangle u_1 + \langle f, u_2\rangle u_2 + \ldots + \langle f, u_n\rangle u_n = \sum_{k=1}^{n} < f, u_k > u_k .$$

....

In the following steps use the inner product space axioms to prove the following claims mentioned above:

2a) Define $\| f \| = \sqrt{\langle f, f\rangle}$. Show that if $\langle f, g\rangle = 0$, then the Pythagorean Theorem holds for the triangle with sides $f, g, f + g$:

$$\| f + g \|^2 = \|f\|^2 + \| g \|^2 .$$

So we define f, g to be orthogonal or perpendicular whenever $\langle f, g\rangle = 0$. And we say that $(f, g, f + g)$ form a right triangle with hypotenuse $f + g$.

2b) For any two non-zero vectors f, g define

$$proj_g f = \left\langle f, \frac{g}{\| g \|}\right\rangle \frac{g}{\| g \|}, \quad h = f - proj_g f.$$

Show that $(proj_g f, h, f)$ is a right triangle with hypotenuse f. Conclude that

$$\left\| proj_g f \right\| \leq \| f \|$$

with equality if and only if $proj_g f = f$, i.e. f, g are positive scalar multiples of each other. Deduce the Cauchy-Schwarz inequality

$$\langle f, g\rangle \leq \| f \| \ \| g \|$$

with equality if and only if f, g are positive scalar multiples of each other. (This calculation also lets us define angles between vectors in inner product spaces:
In analogy with the situation in Euclidean right triangles with the same lengths, we define the angle θ between f, g to be given by

$$\cos(\theta) = \frac{\left\langle f, \frac{g}{\| g \|}\right\rangle}{\| f \|} = \frac{\langle f, g\rangle}{\| f \| \ \| g \|} \quad .)$$

2c) Use the Cauchy-Schwarz inequality to prove the triangle inequality

$$\|f + g\| \leq \| f \| + \| g \| .$$

Hint: square both sides and prove that equivalent identity.

2d) Let W be a finite-dimensional subspace of the inner product space V, and let $\{\underline{\boldsymbol{u}}_1, \underline{\boldsymbol{u}}_2, \ldots \underline{\boldsymbol{u}}_n\}$ be an ortho-normal basis for W, i.e. the magnitudes $\|\underline{\boldsymbol{u}}_i\| = 1$, and $\langle \underline{\boldsymbol{u}}_i, \underline{\boldsymbol{u}}_j \rangle = 0$ for $i \neq j$. Show that for any $f \in V$,

$$proj_W f := \langle f, u_1 \rangle u_1 + \langle f, u_2 \rangle u_2 + \ldots + \langle f, u_n \rangle u_n = \sum_{k=1}^{n} < f, u_k > u_k$$

is the nearest element of W to f. Hint: Show that for any $w \in W$, $(f - proj_W f, proj_W f - w, f - w)$ is a right triangle with hypotenuse $f - w$, i.e.

$$\| w - f \|^2 = \|f - proj_W f\|^2 + \|proj_W f - w\|^2.$$

(One can find ortho-normal bases using the Gram-Schmidt algorithm that you learned in Math 2270 ... Wikipedia has a page on Gram Schmidt and projection for inner product spaces, if you're interested. For our case of Fourier series the orthonormal functions were handed to us.)