• next week's notes are posted
• quiz today.

Math 2280-001
Fri Apr 7

7.1-7.4 Laplace transform, and application to DE IVPs, especially those in Chapter 3. Today we'll continue to fill in the Laplace transform table (at the end of the notes). Along the way we'll revisit some of the mechanical oscillation differential equations from Chapter 3.

$f(t)$	$F(s)$
1	$1/s$
e^{at}	$1/(s-a)$
$c_1 f_1 + c_2 f_2$	$c_1 F_1 + c_2 F_2$

Exercise 1) (to review) Use the table to compute

1a) $\mathcal{L}\{4 + 2e^{-4t}\}(s)$

1b) $\mathcal{L}^{-1}\left\{\frac{2}{s-2} + \frac{6}{s}\right\}(t)$.

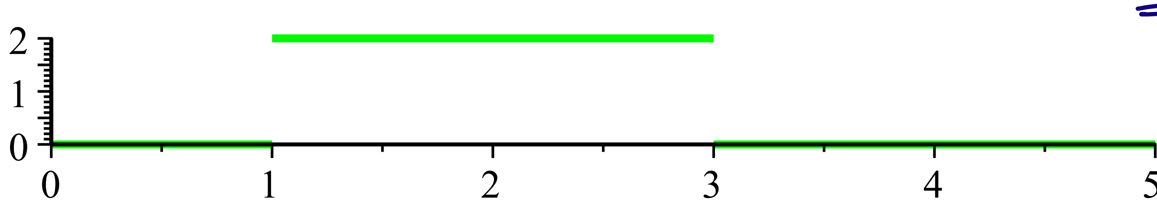
1a) $\frac{4}{s} + 2\frac{1}{s+4}$

1b) $2e^{2t} + 6$

Exercise 2) (to review the definition) Use the definition of Laplace transform,

$$F(s) = \mathcal{L}\{f(t)\}(s) := \int_0^\infty f(t)e^{-st}\,dt$$

to find the Laplace transform of the step function graphed below. (The function is equal to zero for $t \geq 3$.)



$$\int_0^\infty f(t)e^{-st}\,dt = \int_0^1 + \int_1^3 + \int_3^\infty \quad (\text{first and last} \to 0)$$

$$f(t) = \begin{cases} 0 & 0 \le t < 1 \\ 2 & 1 \le t < 3 \\ 0 & t \geq 3 \end{cases}$$

$$= \int_1^3 2e^{-st}\,dt = 2\frac{e^{-st}}{-s}\Big]_{t=1}^3 = -\frac{2}{s}e^{-3s} + \frac{2}{s}e^{-s}$$

Exercise 3a) Use the Table entry we proved on Wednesday for derivatives (via integration by parts), namely

$$* \quad \mathcal{L}\{g'(t)\}(s) = s\,\mathcal{L}\{g(t)\}(s) - g(0) = s\,G(s) - g(0)$$

and math induction to show that for $n \in \mathbb{N}$

$$\mathcal{L}\{f^{(n)}(t)\}(s) = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - s f^{(n-2)}(0) - f^{(n-1)}(0).$$

$n=2: \ \mathcal{L}\{f''(t)\}(s) = s^2 F(s) - s f(0) - f'(0)$

- know for $n=1$ $*$ on Wed.
- Show if it's true for $n=k$ then it's true for $n=k+1$

} conclude: true $\forall\, n = 1, 2, \ldots$ $n \in \mathbb{N}$

Assume $\mathcal{L}\{f^{(k)}(t)\}(s) = s^k \mathcal{L}\{f(t)\}(s) - s^{k-1} f(0) - \ldots - f^{(k-1)}(0)$

Apply $*$ with $g(t) = f^{(k)}(t)$, $g'(t) = f^{(k+1)}(t)$

$* \Rightarrow \mathcal{L}\{f^{(k+1)}(t)\} = s\,\mathcal{L}\{f^{(k)}(t)\}(s) - f^{(k)}(0)$

$= s^{k+1} F(s) - s^k f(0) - s^{k-1} f'(0) - \ldots - s f^{(k-1)}(0) - f^{(k)}(0)$

■

3b) (Integrals are "negative" derivatives): Use the Laplace transform first-derivative formula above to show that

$g'(t) = f(t)$, $g(0) = 0$

$$\mathcal{L}\left\{\int_0^t f(\tau)\, d\tau\right\}(s) = \frac{1}{s}\mathcal{L}\{f(t)\}(s) = \frac{F(s)}{s} \qquad \left(\int_0^t f(\tau)\,d\tau =: g(t)\right)$$

$$(2) \quad \mathcal{L}\left\{\int_0^t \left(\int_0^r f(\tau)\, d\tau\right) dr\right\}(s) = \frac{F(s)}{s^2} \ \ldots$$

$* \quad \mathcal{L}\{f(t)\}(s) = s\,G(s) - g(0)$, with $g(0) \to 0$ and $f(t) = g'(t)$

$F(s) = s\,G(s)$ $\qquad \boxed{G(s) = \frac{F(s)}{s}}$

(2) $g(t) = \int_0^t \left(\int_0^r f(\tau)\,d\tau\right) dr$

$\Rightarrow g'(t) = \int_0^t f(\tau)\,d\tau$

$g''(t) = f(t)$

$\mathcal{L}\{g''(t)\}(s) = s^2 G(s) - s\,g(0) - g'(0)$ $\qquad g(0) = 0, \quad g'(0) = 0$

$F(s) = s^2 G(s) \Rightarrow \boxed{G(s) = \frac{F(s)}{s^2}}$

Exercise 4) Use the result of 3a to show that for $n \in \mathbb{N}$,

$n=1$ $\quad \mathcal{L}\{t\}(s) = \frac{1}{s^2}$

$n=2$ $\quad \mathcal{L}\{t^2\}(s) = \frac{2}{s^3}$

$$\mathcal{L}\{t^n\}(s) = \frac{n!}{s^{n+1}}.$$

$f(t) = t^n$

$f'(t) = n t^{n-1}$

$f''(t) = n(n-1) t^{n-2}$

$\vdots$

$f^{(n)} = n(n-1)\cdots 3\cdot 2\cdot 1$

$f(t) = t^n$

$f^{(n)}(t) = n!$

so $\mathcal{L}\{n!\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$

$\frac{n!}{s} = s^n F(s)$

$F(s) = \frac{n!}{s^{n+1}}$

$f(t)$	$F(s)$
1	$1/s$
$\sin kt$	$\frac{k}{s^2+k^2}$
$\cos kt$	$\frac{s}{s^2+k^2}$

Exercise 5) Find $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\}(t)$

a) using the result of 3b.
b) using partial fractions.

a) $\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\}(s) = \frac{F(s)}{s}$

$F(s) = \frac{1}{s^2+4} = \frac{1}{2}\frac{2}{s^2+4}$

$f(t) = \frac{1}{2}\sin 2t$

$\int_0^t \frac{1}{2}\sin 2\tau\,d\tau = -\frac{1}{4}\cos 2\tau \Big]_0^t$

$= -\frac{1}{4}\cos 2t + \frac{1}{4}$

$\mathcal{L}\left\{-\frac{1}{4}\cos 2t + \frac{1}{4}\right\}(s) = \frac{1}{s(s^2+4)}$

b) $\frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}$

$1 = A(s^2+4) + (Bs+C)s = 4A + s(C) + s^2(A+B)$

$+0s$
$+0s^2$

$4A = 1$
$C = 0$
$A+B = 0$

$4A = 1 \quad A = 1/4$
$B = -1/4$
$C = 0$

$\frac{1}{s(s^2+4)} = \frac{1}{4}\frac{1}{s} - \frac{1}{4}\frac{s}{s^2+4}$

$\mathcal{L}^{-1}:\ \frac{1}{4} - \frac{1}{4}\cos 2t$

Exercise 6) (first translation theorem). Use the definition of Laplace transform to show that

$$\mathcal{L}\{ e^{a\,t} f(t)\}(s) = \mathcal{L}\{f(t)\}(s-a) = F(s-a)$$

$$= \int_0^\infty e^{at} f(t)\, e^{-st}\, dt$$

$$= \int_0^\infty f(t)\, e^{-(s-a)t}\, dt$$

$$= \mathcal{L}\{f(t)\}(s-a)$$

Exercise 7) As a special case of Exercise 6, show

$$\mathcal{L}\{t\, e^{a\,t}\}(s) = \frac{1}{(s-a)^2}.$$

$$\mathcal{L}\{t^n\, e^{a\,t}\}(s) = \frac{n!}{(s-a)^{n+1}}$$

A harder table entry to understand is the following one - go through this computation and see why it seems reasonable, even though there's one step that we don't completely justify. The table entry is

$t f(t)$	$-F'(s)$

$$\frac{d}{ds}\sum_{i=1}^{n} f(t_i)\, e^{-st_i}\, \Delta t_i = \sum_{i=1}^{n} \frac{d}{ds} f(t_i)\, e^{-st_i}\, \Delta t_i$$

Here's how we get it:

$$F(s) = \mathcal{L}\{f(t)\}(s) := \int_0^\infty f(t)e^{-s\,t}\, dt$$

$$\Rightarrow \frac{d}{ds}F(s) = \frac{d}{ds}\int_0^\infty f(t)e^{-s\,t}\, dt = \int_0^\infty \frac{d}{ds} f(t)e^{-s\,t}\, dt\,.$$

It's this last step which is true, but needs more justification. We know that the derivative of a sum is the sum of the derivatives, and the integral is a limit of Riemann sums, so this step does at least seem reasonable. The rest is straightforward:

$$\int_0^\infty \frac{d}{ds} f(t)e^{-s\,t}\, dt = \int_0^\infty f(t)(-t)e^{-s\,t}\, dt = -\mathcal{L}\{t\, f(t)\}(s) \qquad \square.$$

For resonance and other applications ...

Exercise 8) Use $\mathcal{L}\{t f(t)\}(s) = -F'(s)$ directly, or Euler's formula and $\mathcal{L}\{t e^{\alpha t}\}(s) = \frac{1}{(s-\alpha)^2}$ to show

a) $\mathcal{L}\{t\cos(k t)\}(s) = \frac{s^2 - k^2}{(s^2 + k^2)^2}$ ✓

b) $\mathcal{L}\left\{\frac{1}{2k} t\sin(k t)\right\}(s) = \frac{s}{(s^2 + k^2)^2}$ ✓

c) Then use a and the identity

$$\frac{1}{(s^2 + k^2)^2} = \frac{1}{2k^2}\left(\frac{s^2 + k^2}{(s^2 + k^2)^2} - \frac{s^2 - k^2}{(s^2 + k^2)^2}\right)$$

to verify the table entry

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2 + k^2)^2}\right\}(t) = \frac{1}{2k^2}\left(\frac{1}{k}\sin(k t) - t\cos(k t)\right).$$

a) $\mathcal{L}\{t\cos kt\}(s) = -\frac{d}{ds}\mathcal{L}\{\cos kt\}(s) = -\frac{d}{ds}\frac{s}{s^2+k^2}$

$$= -\frac{1\cdot(s^2+k^2) - s(2s)}{(s^2+k^2)^2}$$

$$= \frac{s^2-k^2}{(s^2+k^2)^2}$$

Exercise 9) Use Laplace transforms to write down the solution to

$$x''(t) + \omega_0^2 x(t) = \frac{F_0}{m} \sin(\omega_0 t)$$
$$x(0) = x_0$$
$$x'(0) = v_0 .$$

what phenomena do solutions to this DE illustrate (even though we're forcing with $\sin(\omega_0 t)$ rather than $\cos(\omega_0 t)$)? How would you have tried to solve this problem in Chapter 3?

$$\mathcal{L}: \quad s^2 X(s) - s x_0 - v_0 + \omega_0^2 X(s) = \frac{F_0}{m} \frac{\omega_0}{s^2+\omega_0^2}$$

$$X(s)(s^2+\omega_0^2) = \frac{F_0}{m} \frac{\omega_0}{s^2+\omega_0^2} + x_0 s + v_0$$

$$X(s) = \frac{F_0 \omega_0}{m} \frac{1}{(s^2+\omega_0^2)^2} + x_0 \frac{s}{s^2+\omega_0^2} + v_0 \frac{1}{s^2+\omega_0^2}$$

$$x(t) = \frac{F_0 \omega_0}{m} \frac{1}{2\omega_0^2}\left(\frac{1}{\omega_0} \sin\omega_0 t - t\cos\omega_0 t\right) + x_0 \cos\omega_0 t + \frac{v_0}{\omega_0} \sin\omega_0 t$$

Exercise 10) Solve the following IVP. Use this example to recall the general partial fractions algorithm.

$$x''(t) + 4x(t) = 8te^{2t}$$
$$x(0) = 0$$
$$x'(0) = 1$$

Wolfram checks:

WolframAlpha computational knowledge engine.

x"(t)+4*x(t)=8*t*exp(2*t),x(0)=0, x'(0)=1

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Input:

$$\{x''(t) + 4\,x(t) = 8\,t\,\exp(2\,t),\ x(0) = 0,\ x'(0) = 1\}$$

Open code

ODE classification:

second-order linear ordinary differential equation

Alternate form:

$$\{x''(t) = 8\,e^{2t}\,t - 4\,x(t),\ x(0) = 0,\ x'(0) = 1\}$$

Differential equation solution:

Approximate form Step-by-step solution

$$x(t) = \frac{1}{2}\left(e^{2t}\,(2\,t - 1) + \sin(2\,t) + \cos(2\,t)\right)$$

WolframAlpha computational knowledge engine.

partial fractions

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Assuming "partial fractions" refers to a computation | Use as a general topic instead

rational function: 8/((s-2)^2*(s^2+4))+1

Input:

partial fractions	$\frac{8}{(s-2)^2\,(s^2+4)} + \frac{1}{s^2+4}$

Open code

Result:

Step-by-step solution

$$\frac{8}{(s-2)^2\,(s^2+4)} + \frac{1}{s^2+4} = \frac{s+2}{2\,(s^2+4)} - \frac{1}{2\,(s-2)} + \frac{1}{(s-2)^2}$$

WolframAlpha computational knowledge engine.

inverse laplace transform

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Assuming "inverse laplace transform" refers to a computation | Use as referring to a mathematical definition instead

- function to transform: (1/2)s/(s^2+4)+1/(s^2
- initial variable: s
- transform variable: t

Input:

$$\mathcal{L}_s^{-1}\left[\frac{1}{2} \times \frac{s}{s^2+4} + \frac{1}{s^2+4} - \frac{1}{2} \times \frac{1}{s-2} + \frac{1}{(s-2)^2}\right](t)$$

Open code

$\mathcal{L}_s^{-1}[f(s)](t)$ is the inverse Laplace transform of $f(s)$ with real variable t

Result:

$$\frac{1}{2}\left(2\, e^{2t}\, t - e^{2t} + \sin(2\, t) + \cos(2\, t)\right)$$

$f(t)$, with $\lvert f(t)\rvert \le Ce^{Mt}$	$F(s) := \int_0^\infty f(t)e^{-st}\,dt$ for $s > M$	↓ verified
$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$	☒
1 t t^2 $t^n,\ n \in \mathbb{N}$	$\frac{1}{s}\quad (s > 0)$ $\frac{1}{s^2}$ $\frac{2}{s^3}$ $\frac{n!}{s^{n+1}}$	☒
$e^{\alpha t}$	$\frac{1}{s-\alpha}\quad (s > \Re(\alpha))$	☒
$\cos(k\,t)$ $\sin(k\,t)$ $\cosh(k\,t)$ $\sinh(k\,t)$ $e^{a\,t}\cos(k\,t)$ $e^{a\,t}\sin(k\,t)$	$\frac{s}{s^2+k^2}\quad (s > 0)$ $\frac{k}{s^2+k^2}\quad (s > 0)$ $\frac{s}{s^2-k^2}\quad (s > k)$ $\frac{k}{s^2-k^2}\quad (s > k)$ $\frac{(s-a)}{(s-a)^2+k^2}\quad (s > a)$ $\frac{k}{(s-a)^2+k^2}\quad (s > a)$	☒ ☒ ☒ ☒
$f'(t)$ $f''(t)$ $f^{(n)}(t),\ n \in \mathbb{N}$ $\int_0^t f(\tau)\,d\tau$	$s\,F(s) - f(0)$ $s^2F(s) - s\,f(0) - f'(0)$ $s^n\,F(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$ $\frac{F(s)}{s}$	☒ ☒ ☒ ☒
$t\,f(t)$ $t^2\,f(t)$ $t^n\,f(t),\ n \in \mathbb{Z}$	$-F'(s)$ $F''(s)$ $(-1)^n\,F^{(n)}(s)$	

$\frac{f(t)}{t}$	$\int_s^{\infty} F(\sigma)\, d\sigma$	
$t \cos(k\, t)$ $\frac{1}{2\, k}\, t \sin(k\, t)$ $\frac{1}{2\, k^3}\,(\sin(k\, t) - k\, t \cos(k\, t))$	$\frac{s^2 - k^2}{(s^2 + k^2)^2}$ $\frac{s}{(s^2 + k^2)^2}$ $\frac{1}{(s^2 + k^2)^2}$	
$e^{a\, t} f(t)$ $t\, e^{a\, t}$ $t^n\, e^{a\, t},\ n \in \mathbb{Z}$	$F(s - a)$ $\frac{1}{(s - a)^2}$ $\frac{n!}{(s - a)^{n+1}}$	

Laplace transform table