Wed Apr 5
7.1-7.2 Laplace transform, and application to DE IVPs

- The Laplace transform is a linear transformation "$\mathcal{L}$" that converts piecewise continuous functions $f(t)$, defined for $t \geq 0$ and with at most exponential growth ($|f(t)| \leq Ce^{Mt}$ for some values of C and M), into functions $F(s)$ defined by the transformation formula

inputs

integrand $| \leq Ce^{Mt}e^{-st} = Ce^{(M-s)t}$

outputs $s > M$ $\rightarrow$ $$F(s) = \mathcal{L}\{f(t)\}(s) := \int_0^\infty f(t)e^{-st}\,dt\,.$$

- Notice that the integral formula for $F(s)$ is only defined for sufficiently large s, and certainly for $s > M$, because as soon as $s > M$ the integrand is decaying exponentially, so the improper integral from $t = 0$ to ∞ converges.
- The convention is to use lower case letters for the input functions and (the same) capital letters for their Laplace transforms, as we did for $f(t)$ and $F(s)$ above. Thus if we called the input function $x(t)$ then we would denote the Laplace transform by $X(s)$.

Taking Laplace transforms seems like a strange thing to do. And yet, the Laplace transform $\mathcal{L}$ is just one example of a collection of useful "integral transforms". $\mathcal{L}$ is especially good for solving IVPs for linear DEs, as we shall see starting today. Other famous transforms - e.g. Fourier series and Fourier transform are extremely important in studying linear differential and partial differential equations. We will discuss Fourier series in about a week. These transforms are also studied in Math 3140, 3150, and in various 5000-level pure and applied math classes.

Exercise 1) Use the definition of Laplace transform

$$F(s) = \mathcal{L}\{f(t)\}(s) := \int_0^\infty f(t)e^{-st}\,dt$$

to check the following facts, which you will also find inside the front cover of your text book.

a) $\mathcal{L}\{1\}(s) = \dfrac{1}{s} \quad (s > 0)$

b) $\mathcal{L}\{e^{\alpha t}\}(s) = \dfrac{1}{s-\alpha} \quad (s > \alpha \text{ if } \alpha \in \mathbb{R},\ s > a \text{ if } \alpha = a + ki \in \mathbb{C})$

c) Laplace transform is linear, i.e.

$$\mathcal{L}\{f_1(t) + f_2(t)\}(s) = F_1(s) + F_2(s)\,.$$
$$\mathcal{L}\{cf(t)\}(s) = cF(s).$$

d) Use linearity and your work above to compute $\mathcal{L}\{3 - 4e^{-2t}\}(s)$.

a) $\mathcal{L}\{1\} = \int_0^\infty 1\, e^{-st}\,dt = \dfrac{e^{-st}}{-s}\Big]_{t=0}^{\infty} = 0 - \left(\dfrac{1}{-s}\right) = \dfrac{1}{s} \quad (s>0)$

b) $\mathcal{L}\{e^{\alpha t}\}(s) = \int_0^\infty e^{\alpha t}e^{-st}\,dt = \int_0^\infty e^{(\alpha-s)t}\,dt = \dfrac{e^{(\alpha-s)t}}{\alpha-s}\Big]_{t=0}^{\infty} = 0 - \dfrac{1}{\alpha-s} = \dfrac{1}{s-\alpha}$ $(s>\alpha)$ $(s > \text{Real}(\alpha))$

c) $\mathcal{L}\{c_1f_1(t) + c_2f_2(t)\}(s) = \int_0^\infty (c_1f_1(t) + c_2f_2(t))e^{-st}\,dt$
$= c_1F_1(s) + c_2F_2(s)$ because integration is linear

d) $\mathcal{L}\{3 - 4e^{-2t}\}(s)$
$= 3\dfrac{1}{s} - 4\dfrac{1}{s+2}$

For the linear differential equations and systems of differential equations that we've been studying, the following Laplace transforms are very important:

Exercise 2) Use complex number algebra, including Euler's formula, linearity, and the result from 1b that

$$\mathcal{L}\{e^{(a+ki)t}\}(s) = \frac{1}{s-(a+ki)}$$

to verify that

a) $\mathcal{L}\{\cos(kt)\}(s) = \dfrac{s}{s^2+k^2}$

b) $\mathcal{L}\{\sin(kt)\}(s) = \dfrac{k}{s^2+k^2}$

c) $\mathcal{L}\{e^{at}\cos(kt)\}(s) = \dfrac{s-a}{(s-a)^2+k^2}$ $*$

d) $\mathcal{L}\{e^{at}\sin(kt)\}(s) = \dfrac{k}{(s-a)^2+k^2}$. Δ

a=0

(Notice that if we tried doing these Laplace transforms directly from the definition, the integrals would be messy but we could attack them via integration by parts or integral tables.)

$$\mathcal{L}\{e^{(a+ik)t}\}(s) = \frac{1}{s-(a+ik)}$$

$$\mathcal{L}\{e^{at}\cos kt + i e^{at}\sin kt\}(s) = \text{''} \quad \frac{1}{(s-a)-ik}\cdot\frac{(s-a)+ik}{(s-a)+ik}$$

$$\underbrace{\mathcal{L}\{e^{at}\cos kt\}(s)}_{*} + i\underbrace{\mathcal{L}\{e^{at}\sin kt\}(s)}_{\Delta} = \frac{(s-a)+ik}{(s-a)^2+k^2} = \underbrace{\frac{s-a}{(s-a)^2+k^2}}_{*} + i\underbrace{\frac{k}{(s-a)^2+k^2}}_{\Delta}$$

It's a theorem (hard to prove but true) that a given Laplace transform $F(s)$ can arise from at most one piecewise continuous function $f(t)$. (Well, except that the values of f at the points of discontinuity can be arbitrary, as they don't affect the integral used to compute $F(s)$.) Therefore you can read Laplace transform tables in either direction, i.e. not only to deduce Laplace transforms, but inverse Laplace transforms $\mathcal{L}^{-1}\{F(s)\}(t) = f(t)$ as well.

Exercise 3) Use the Laplace transforms we've computed and linearity to compute

$$\mathcal{L}^{-1}\left\{\frac{7}{s} + \frac{1}{s^2+16} - \frac{10\,s}{s^2+16}\right\}(t)\,.$$

$$= 7 + \frac{1}{4}\sin 4t - 10\cos 4t$$

$\mathcal{L}$ is 1-1.
only fn $\to 0$ is $f=0$
proof is "fun"
"Stone-Weierstrasse Thm"

$f(t)$ $\lvert f(t)\rvert \le Ce^{Mt}$	$F(s) := \int_0^\infty f(t)e^{-st}\,dt$ for $s > M$
$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$
1	$\frac{1}{s} \quad (s > 0)$
$e^{\alpha t}$	$\frac{1}{s-\alpha} \quad (s > \Re(a))$
$\cos(k\,t)$	$\frac{s}{s^2+k^2} \quad (s > 0)$
$\frac{1}{k}\sin kt$ $\quad \sin(k\,t)$	$\frac{1}{s^2+k^2}$ $\quad \frac{k}{s^2+k^2} \quad (s > 0)$
$e^{a\,t}\cos(k\,t)$	$\frac{(s-a)}{(s-a)^2+k^2} \quad (s > a)$
$e^{a\,t}\sin(k\,t)$	$\frac{k}{(s-a)^2+k^2} \quad (s > a)$
$f'(t)$	$s\,F(s) - f(0)$
$f''(t)$	$s^2F(s) - s\,f(0) - f'(0)$
$f'''(t)$	$s^3F(s) - s^2 f(0) - s\,f'(0) - f''(0)$

$\frac{1}{4}\,\frac{4}{s^2+16} \qquad k=4$

Laplace transform table

The integral transforms of DE's and PDE's were designed to have the property that they convert the corresponding linear DE and PDE problems into algebra problems. For the Laplace transform it's because of these facts:

Exercise 4a) Use integration by parts and the definition of Laplace transform to show that

$$\mathcal{L}\{g'(t)\}(s) = s\,\mathcal{L}\{g(t)\}(s) - g(0) = s\,G(s) - g(0).$$

4b) Use the result of a, applied to the function $f'(t)$ to show that

$$\mathcal{L}\{f''(t)\}(s) = s^2F(s) - s\,f(0) - f'(0).$$

4c) What would you guess is the Laplace transform of $f'''(t)$? Could you check this?

4a) $\mathcal{L}\{g'(t)\}(s) = \int_0^\infty g'(t)e^{-st}\,dt = e^{-st}g(t)\Big]_{t=0}^{\infty} - \int_0^\infty g(t)(-s)e^{-st}\,dt$

$u = e^{-st}$ $\quad du = -se^{-st}dt$

$dv = g'(t)dt$ $\quad v = g(t)$

$s>M$: $= 0 - g(0) + s\int_0^\infty g(t)e^{-st}\,dt$

$|g(t)| \le Ce^{Mt}$ some C, M.

$= s\,G(s) - g(0)$

4b) use 4a) with $g(t) = f'(t)$

$\mathcal{L}\{f''(t)\}(s) = s\,\mathcal{L}\{f'(t)\}(s) - f'(0) = s\left[s\,F(s) - f(0)\right] - f'(0)$

$= s^2F(s) - s\,f(0) - f'(0)$

Here's an example of using Laplace transforms to solve DE IVPs, in the context of Chapter 3 and the mechanical (and electrical) application problems we considered there.

Exercise 5) Consider the undamped forced oscillation IVP

$$x''(t) + 4x(t) = 10\cos(3t)$$
$$x(0) = 2$$
$$x'(0) = 1$$

If $x(t)$ is the solution, then both sides of the DE are equal. Thus the Laplace transforms are equal as well... .so, equate the Laplace transforms of each side and use algebra to find $\mathcal{L}\{x(t)\}(s) = X(s)$. Notice you've computed $X(s)$ without actually knowing $x(t)$! If you were happy to stay in "Laplace land" you'd be done. In any case, at this point you can use our table entries to find $x(t) = \mathcal{L}^{-1}\{x(t)\}(s)$.

(Notice that if your algebra skills are good you've avoided having to use the Chapter 3 algorithm of (i) find x_H (ii) find an x_P (iii) $x = x_P + x_H$ (iv) solve IVP.) Magic! Or, would you have prefered to convert to a first order system and to have used variation of parameters with an FM or e^{At}, like in Chapter 5 :-) ?)

for sol'n $x(t)$, $\mathcal{L}(LHS) = \mathcal{L}(RHS)$ because LHS = RHS

$s^2X(s) - 2s - 1 + 4X(s) = 10\dfrac{s}{s^2+9}$

$X(s)(s^2+4) = 10\dfrac{s}{s^2+9} + 2s + 1$

$X(s) = 10\dfrac{s}{(s^2+4)(s^2+9)} + \dfrac{2s}{s^2+4} + \dfrac{1}{s^2+4}$

$f(t)$	$F(s)$
x''	$s^2X(s) - s\,x(0) - x'(0)$
$\cos(kt)$	$\dfrac{s}{s^2+k^2}$
$\sin kt$	$\dfrac{k}{s^2+k^2}$

"done" but not really

$$X(s) = 10s\,\frac{1}{(s^2+4)(s^2+9)} + \frac{2s}{s^2+4} + \frac{1}{s^2+4}$$

$$= \cancel{10}s\,\frac{\overset{2}{1}}{\cancel{5}}\left[\frac{1}{s^2+4} - \frac{1}{s^2+9}\right] + \frac{2s}{s^2+4} + \frac{1}{s^2+4}. \qquad \frac{\cancel{s^2}+9-(\cancel{s^2}+4)}{(s^2+4)(s^2+9)}$$

$$X(s) = 4\,\frac{s}{s^2+4} - \frac{2s}{s^2+9} + \frac{1}{2}\,\frac{2}{s^2+4}$$

$$x(t) = 4\cos 2t - 2\cos 3t + \tfrac{1}{2}\sin 2t$$

$$= \underbrace{-2\cos 3t}_{\text{“}x_p\text{”}} + \underbrace{4\cos 2t}_{\text{“}x_H\text{”}} + \tfrac{1}{2}\sin 2t$$

Input:

$\{x''(t) + 4\,x(t) = 10\cos(3\,t),\ x(0) = 2,\ x'(0) = 1\}$

Open code

ODE classification:

second-order linear ordinary differential equation

Alternate forms: More

$\{x''(t) = 10\cos(3\,t) - 4\,x(t),\ x(0) = 2,\ x'(0) = 1\}$

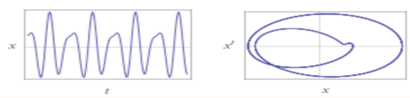
$\{x''(t) + 4\,x(t) = 10\cos(t)\,(2\cos(2\,t) - 1),\ x(0) = 2,\ x'(0) = 1\}$

$\{x''(t) + 4\,x(t) = 10\cos^3(t) - 30\cos(t)\sin^2(t),\ x(0) = 2,\ x'(0) = 1\}$

Differential equation solution: Step-by-step solution

$$x(t) = \frac{1}{2}\,(\sin(2\,t) + 8\cos(2\,t) - 4\cos(3\,t))$$

Plots of the solution:



Enlarge | Data | Customize | Plaintext | Interactive

Input:

partial fractions	$10 \times \frac{s}{(s^2+9)(s^2+4)} + \frac{2\,s+1}{s^2+4}$

Open code

Result: Step-by-step solution

$$\frac{10\,s}{(s^2+4)(s^2+9)} + \frac{2\,s+1}{s^2+4} = \frac{4\,s+1}{s^2+4} - \frac{2\,s}{s^2+9}$$

inverse Laplace transform

Web Apps   Examples   Random

Assuming "inverse Laplace transform" refers to a computation | Use as referring to a mathematical definition instead

- function to transform: (4*s+1)/(s^2+4)-2s/(s
- initial variable: s
- transform variable: t

Input:

$$\mathcal{L}_s^{-1}\left[\frac{4\,s+1}{s^2+4} - 2 \times \frac{s}{s^2+9}\right](t)$$

Open code

$\mathcal{L}_s^{-1}[f(s)](t)$ is the inverse Laplace transform of $f(s)$ with real variable t

Result:

$$\frac{1}{2}\,(\sin(2\,t) + 8\cos(2\,t) - 4\cos(3\,t))$$

Exercise 6) Use Laplace transform as above, to solve the IVP for the following underdamped, unforced oscillator DE:

$$x''(t) + 6x'(t) + 34x(t) = 0$$
$$x(0) = 3$$
$$x'(0) = 1$$

x'	$sX(s) - x(0)$
x''	$s^2X(s) - sx(0) - x'(0)$

$$s^2X(s) - s\underbrace{x(0)}_{=3} - \underbrace{x'(0)}_{=1} + 6\left(sX(s) - \underbrace{x(0)}_{=3}\right) + 34X(s) = 0$$

$$X(s)\left[s^2 + 6s + 34\right] = 3s + 19$$

$$X(s) = \frac{3s+19}{s^2+6s+34}$$ "done"

$$s^2 + 6s + 34 = (s+3)^2 + 25$$

$e^{at}\cos kt$	$\frac{s-a}{(s-a)^2+k^2}$
$e^{at}\sin kt$	$\frac{k}{(s-a)^2+k^2}$

$k = 5$
$a = -3$

$$X(s) = \frac{3(s+3)+10}{(s+3)^2+25}$$

$$X(s) = 3\frac{s+3}{(s+3)^2+25} + 2\frac{\cancel{10}\ 5}{(s+3)^2+25}$$

$$x(t) = 3e^{-3t}\cos 5t + 2e^{-3t}\sin 5t$$