Math 2280-001
Week 12, April 3-7

Mon Apr 3   Finish section 5.7 notes from ~~Friday~~ Wednesday. We will also discuss questions you may have about the section 5.6 homework due on Tuesday. A lab problem for next week will be to work the input-output problem from the second midterm using (1) matrix exponentials, and (2) the diagonalization method. We may begin that problem in lab format, if time permits.

finish up day
handout today for lab problem tomorrow

$$\begin{bmatrix} -t\,e^{2t} \\ (1-t)\,e^{2t} \end{bmatrix}$$

$$\begin{bmatrix} e^{2t}(1+t) & -t\,e^{2t} \\ t\,e^{2t} & (1-t)\,e^{2t} \end{bmatrix}$$ $\Phi(t)$, $\Phi(0) = I$ in this case

$$\begin{bmatrix} e^{2t}(1+t) & -t\,e^{2t} \\ t\,e^{2t} & (1-t)\,e^{2t} \end{bmatrix}$$ $\Phi(t)\,\Phi(0)^{-1}$

$$\begin{bmatrix} e^{2t}(1+t) & -t\,e^{2t} \\ t\,e^{2t} & -e^{2t}(-1+t) \end{bmatrix}$$ e^{tA} **(4)**

>

>

Last Wed we computed e^{tA} for an A not diagonalizable

didn't finish rest of notes.

recall

$$e^{At} = I + tA + \frac{t^2}{2!}A^2 + \dots$$

$$= \Phi(t)\,\Phi(0)^{-1}$$

F.M. , columns basis of sol'ns to $\vec{x}\,' = A\vec{x}$

Start here Monday 4/3

Variation of parameters: This is what fundamental matrices and matrix exponentials are especially good for....they let you solve non-homogeneous systems without guessing. Consider the non-homogeneous first order system

$$\underline{\boldsymbol{x}}'(t) = P(t)\underline{\boldsymbol{x}} + \underline{\boldsymbol{f}}(t) \quad *$$

Let $\Phi(t)$ be an FM for the homogeneous system

$$\underline{\boldsymbol{x}}'(t) = P(t)\underline{\boldsymbol{x}}.$$

$\Phi'(t) = P(t)\Phi(t)$

$\Phi(0)$ is invertible.

Since $\Phi(t)$ is invertible for all t we may do a change of functions for the non-homogeneous system:

(in fact, $\Phi(t)$ is invertible $\forall t$ on interval)

$$\underline{\boldsymbol{x}}(t) = \Phi(t)\underline{\boldsymbol{u}}(t)$$

plug into the non-homogeneous system (*):

$$\cancel{\Phi'(t)\underline{\boldsymbol{u}}(t)} + \Phi(t)\underline{\boldsymbol{u}}'(t) = \cancel{P(t)\Phi(t)\underline{\boldsymbol{u}}(t)} + \underline{\boldsymbol{f}}(t).$$

Since $\Phi' = P\,\Phi$ the first terms on each side cancel each other and we are left with

$$\Phi(t)\underline{\boldsymbol{u}}'(t) = \underline{\boldsymbol{f}}(t)$$

$$\underline{\boldsymbol{u}}' = \Phi^{-1}\underline{\boldsymbol{f}} \quad \leftarrow \text{RHS is just a fn of } t$$

which we can integrate to find a $\underline{\boldsymbol{u}}(t)$, hence an $\underline{\boldsymbol{x}}(t) = \Phi(t)\underline{\boldsymbol{u}}(t)$.

Remark: This is where the (mysterious at the time) formula for variation of parameters in n^{th} order linear DE's came from....

"Recall" (February 24 notes):

....

Variation of Parameters: The advantage of this method is that is always provides a particular solution, even for non-homogeneous problems in which the right-hand side doesn't fit into a nice finite dimensional subspace preserved by L, and even if the linear operator L is not constant-coefficient. The formula for the particular solutions can be somewhat messy to work with, however, once you start computing.

Here's the formula: Let $y_1(x), y_2(x), \ldots y_n(x)$ be a basis of solutions to the homogeneous DE

$$L(y) := y^{(n)} + p_{n-1}(x)y^{(n-1)} + \ldots + p_1(x)y' + p_0(x)y = 0\,. \leftarrow$$

Then $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) + \ldots + u_n(x)y_n(x)$ is a particular solution to

$$L(y) = f$$

provided the coefficient functions (aka "varying parameters") $u_1(x), u_2(x), \ldots u_n(x)$ have derivatives satisfying the Wronskian matrix equation

$$\underbrace{\begin{bmatrix} y_1 & y_2 & \ldots & y_n \\ y_1' & y_2' & \ldots & y_n' \\ \ldots & \ldots & \ldots & \ldots \\ y_1^{(n-1)} & y_2^{(n-1)} & \ldots & y_n^{(n-1)} \end{bmatrix}}_{W(y_1, y_2, \ldots y_n)} \begin{bmatrix} u_1' \\ u_2' \\ : \\ u_n' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ : \\ f \end{bmatrix}$$

Chapter 3

....

$$y'' + x^2 y' + e^x y = 0$$

But if we convert the n^{th} order DE into a first order system for $x_1 = y, x_2 = y'$ etc. we have

$$y^{(n)} + p_{n-1} y^{(n-1)} + \dots + p_1 y' + p_0 y = f \qquad (1)$$

$$\begin{aligned}
x_1 \quad &(= y) \\
x_1{}' = x_2 \quad &(= y') \\
x_2{}' = x_3 \quad &(= y'') \\
x_{n-1}{}' = x_n \quad &\left(= y^{(n-1)}\right) \\
x_n{}' = \left(= y^{(n)}\right) = -p_0(x)x_1 - p_1(x)x_2 - &\dots - p_{n-1}(x)x_{n-1} + f.
\end{aligned}$$

And each basis solution $y(t)$ for $L(y) = 0$ gives a solution $\left[y, y', y'', \dots y^{(n-1)}\right]^T$ to the homogeneous system

$$\begin{bmatrix} x_1{}' \\ x_2{}' \\ x_3{}' \\ \vdots \\ x_n{}' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & 1 \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -p_0 & -p_1 & -p_2 & \dots & -p_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ f \end{bmatrix}. \qquad (2)$$

So the original Wronskian matrix for the n^{th} order linear homogeneous DE is a FM for the system above, so the formula we learned in Chapter 3 is a special case of the easier to understand one for first order systems that we just derived, namely

$$\Phi(t)\underline{\boldsymbol{u}}'(t) = \boldsymbol{f}(t)$$
$$\underline{\boldsymbol{u}}' = \Phi^{-1}\boldsymbol{f}$$

Chapter 5

(1)

$$\begin{bmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & & y_n^{(n-1)} \end{bmatrix}$$

$=$

also Wronskian for (2)
has basis of solns to $\vec{x}' = A\vec{x}$
in columns.

$=$

$\Phi(t)$

If $y(t)$ solves (1)
then $\begin{bmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{bmatrix}$ solves (2).

(if $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ solves (2),
then $x_1(t)$ solves (1))

Returning to first order systems, if we want to solve an IVP for a first order system rather than find the complete general solution, then the following two ways are appropriate:

1) If you want to solve the IVP

$$\underline{\boldsymbol{x}}'(t) = P(t)\underline{\boldsymbol{x}} + \underline{\boldsymbol{f}}(t)$$
$$\underline{\boldsymbol{x}}(0) = \underline{\boldsymbol{x}}_0$$

The the solution will be of the form $\underline{\boldsymbol{x}} = \Phi\,\underline{\boldsymbol{u}}$ (where $\underline{\boldsymbol{u}}' = \Phi^{-1}\underline{\boldsymbol{f}}$ as above). Thus

$$\underline{\boldsymbol{x}}_0 = \Phi(0)\underline{\boldsymbol{u}}_0$$

so

$$\Downarrow.$$

$$\underline{\boldsymbol{u}}_0 = \Phi(0)^{-1}\underline{\boldsymbol{x}}_0.$$

$\vec{u}'(t) = \Phi^{-1}(t)\vec{f}(t)$

Thus

$$\underline{\boldsymbol{u}}(t) = \underline{\boldsymbol{u}}_0 + \int_0^t \underline{\boldsymbol{u}}'(s)\,\mathrm{d}s$$

FTC.

$$\underline{\boldsymbol{u}}(t) = \underline{\boldsymbol{u}}_0 + \int_0^t \boldsymbol{\Phi}^{-1}(s)\underline{\boldsymbol{f}}(s)\,\mathrm{d}s.$$

Then

$$\underline{\boldsymbol{x}}(t) = \Phi(t)\underline{\boldsymbol{u}}(t)$$

$$* \qquad \underline{\boldsymbol{x}}(t) = \Phi(t)\left(\underline{\boldsymbol{u}}_0 + \int_0^t \boldsymbol{\Phi}^{-1}(s)\underline{\boldsymbol{f}}(s)\,\mathrm{d}s\right).$$

special case

$P(t) = A$, const.

$\Phi = e^{tA}$

$\Phi(0) = I$

$*$ becomes $\vec{x}(t) = e^{At}\left(\vec{x}_0 + \int_0^t e^{-sA}\vec{f}(s)\,ds\right)$

2) If you want to solve the special case IVP

$$\begin{cases} \underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}} + \underline{\boldsymbol{f}}(t) \\ \underline{\boldsymbol{x}}(0) = \underline{\boldsymbol{x}}_0 \end{cases}$$

where A is a constant matrix, you may derive a special case of the solution formula above just as we did in Chapter 1. This is sort of amazing!

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}} + \underline{\boldsymbol{f}}(t)$$
$$\underline{\boldsymbol{x}}'(t) - A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{f}}(t)$$

I.F.

$$\mathrm{e}^{-tA}(\underline{\boldsymbol{x}}'(t) - A\,\underline{\boldsymbol{x}}) = \mathrm{e}^{-tA}\underline{\boldsymbol{f}}(t)$$

$$\int_0^t \left(\frac{d}{dt}\left(\mathrm{e}^{-tA}\underline{\boldsymbol{x}}(t)\right) = \mathrm{e}^{-tA}\underline{\boldsymbol{f}}(t) \right).$$

$$\frac{d}{dt}\left(e^{-tA}\vec{x}(t)\right) = e^{-tA}(-A)\vec{x} + e^{-tA}\vec{x}' = e^{-tA}(-A\vec{x} + \vec{x}')$$

Integrate from 0 to t:

$$\mathrm{e}^{-tA}\underline{\boldsymbol{x}}(t) - \underline{\boldsymbol{x}}_0 = \int_0^t \mathrm{e}^{-sA}\underline{\boldsymbol{f}}(s)\,\mathrm{d}s$$

Move the $\underline{\boldsymbol{x}}_0$ over and multiply both sides by e^{tA}:

$$\boxed{\underline{\boldsymbol{x}}(t) = \mathrm{e}^{tA}\left(\underline{\boldsymbol{x}}_0 + \int_0^t \mathrm{e}^{-sA}\underline{\boldsymbol{f}}(s)\,\mathrm{d}s\right).}$$

in theory this formula tells us how to solve any constant coefficient linear DE or system of DE's !

skip (because you'll do the lab problem instead)

Exercise 2 Consider the non-homogeneous problem related to the homogeneous system in Exercise 1:

$$\begin{bmatrix} x_1{}' \\ x_2{}' \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} t \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Solve this system using the formula

$$\underline{\boldsymbol{x}}(t) = \mathrm{e}^{t\,A}\left(\underline{\boldsymbol{x}}_0 + \int_0^t \mathrm{e}^{-s\,A}\underline{\boldsymbol{f}}(s)\;\mathrm{d}s\right)$$

(One could also try undetermined coefficients, but variation of parameters requires no "guessing.")

Tech check: (The commands are sort of strange, but might help in your homework.)

```
> with(LinearAlgebra) :
> A := [3 -1; 1 1] :
  MatrixExponential(t·A);
  f := t -> [t; 0] :
  x0 := [0; 0] :
```

$$\begin{bmatrix} \mathrm{e}^{2t}(1+t) & -t\,\mathrm{e}^{2t} \\ t\,\mathrm{e}^{2t} & -\mathrm{e}^{2t}(-1+t) \end{bmatrix} \qquad (5)$$

```
> integrand := s -> simplify(MatrixExponential(-s·A).f(s)) : #integrand in formula above
> integrand(t); #checking
```

$$\begin{bmatrix} -e^{-2t}(-1+t)\,t \\ -t^2 e^{-2t} \end{bmatrix} \quad (6)$$

```
> integrated := unapply(map(int, integrand(s), s = 0 .. t), t) : #"map" applies a function to each entry of an array...
    # "unapply" makes a function out of output
> integrated(t); #checking
```

$$\begin{bmatrix} \frac{1}{2} t^2 e^{-2t} \\ -\frac{1}{4} + \frac{1}{4} e^{-2t} + \frac{1}{2} t e^{-2t} + \frac{1}{2} t^2 e^{-2t} \end{bmatrix} \quad (7)$$

```
> x := unapply(simplify(MatrixExponential(t·A).(x0 + integrated(t))), t) :
  x(t); #checking answer
```

$$\begin{bmatrix} \frac{1}{4} t\left(e^{2t} - 1\right) \\ \frac{1}{4} e^{2t}(-1+t) + \frac{1}{4} t + \frac{1}{4} \end{bmatrix} \quad (8)$$

```
> with(DEtools) :
  dsolve({x1'(t) = 3·x1(t) - x2(t) + t, x2'(t) = x1(t) + x2(t), x1(0) = 0, x2(0) = 0});
```

$$\left\{x1(t) = \frac{1}{4} t e^{2t} - \frac{1}{4} t,\ x2(t) = -\frac{1}{4} e^{2t} + \frac{1}{4} + \frac{1}{4} t + \frac{1}{4} t e^{2t}\right\} \quad (9)$$

```
>
>
```

Monday 4/3

5.6.39

$$A = \begin{bmatrix} 1 & 3 & 3 & 3 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & 0 \\ & 0 & 2 & \\ & & & 2 \end{bmatrix} + \begin{bmatrix} 0 & 3 & 3 & 3 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= D + N$$

find e^{At}, $\quad e^{tA} = e^{tD + tN}$

$$= e^{tD} e^{tN}$$

I told you this in class, but it's wrong. N & D do not commute so the rule of exponents does not work. Have to resort to generalized eigenspaces

Math 2280-001
Monday April 3

we started working on this in class... it's the 1st lab problem for Tuesday.

In some sense the examples/methods in this problem contain pretty much all of the course so far.

w12.1 (from next week's homework) Consider the input-output IVP from the second midterm

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 20 \\ 40 \end{bmatrix}$$
$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$E_{\lambda=-2} = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$

$E_{\lambda=-5} = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$

w12.1a) Using matrix exponentials, the solution to

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}} + \underline{\boldsymbol{f}}(t)$$
$$\underline{\boldsymbol{x}}(0) = \underline{\boldsymbol{x}}_0$$

is given by

$$\underline{\boldsymbol{x}}(t) = e^{tA}\left(\underline{\boldsymbol{x}}_0 + \int_0^t e^{-sA}\underline{\boldsymbol{f}}(s)\,ds\right).$$

Verify that this recovers the correct solution

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix} - 10\,e^{-2t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

w12.1b) Alternately, when A is diagonalizable, as it is in this case, we can change output variables to reduce to a simpler system: Let

$$A\,S = S\,\Lambda$$

where the columns of S are an eigenbasis for $\mathbb{R}^n$ or $\mathbb{C}^n$ and Λ is the diagonal matrix of corresponding eigenvalues. Then

$$\underline{\mathbf{x}}'(t) = A\,\underline{\mathbf{x}} + \underline{\mathbf{f}}(t)$$
$$\underline{\mathbf{x}}(0) = \underline{\mathbf{x}}_0$$

is equivalent to

$$S^{-1}\underline{\mathbf{x}}' = S^{-1} A\,\underline{\mathbf{x}} + S^{-1}\underline{\mathbf{f}}(t).$$

For

$$\underline{\mathbf{u}}(t) := S^{-1}\underline{\mathbf{x}}(t)$$
$$\underline{\mathbf{u}}'(t) = S^{-1}\underline{\mathbf{x}}'(t)$$

this yields the diagonal system

$$\underline{\mathbf{u}}'(t) = S^{-1}AS\,\underline{\mathbf{u}}(t) + S^{-1}\underline{\mathbf{f}}(t)$$

i.e. the IVP

$$\underline{\mathbf{u}}'(t) = \Lambda\,\underline{\mathbf{u}}(t) + S^{-1}\underline{\mathbf{f}}(t)$$
$$\underline{\mathbf{u}}(0) = S^{-1}\underline{\mathbf{x}}_0$$

After finding $\underline{\mathbf{u}}(t)$, the original solution $\underline{\mathbf{x}}(t) = S\,\underline{\mathbf{u}}(t)$. Rework the tank IVP using this method.