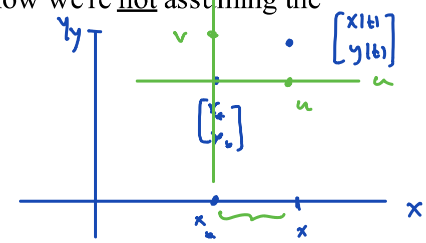
How to linearize with multivariable Calculus: (This would work for systems of n autonomous first order differential equations, but we focus on $n = 2$ in this chapter. Notice how we're not assuming the equilbrium point is the origin. Here's the general system:



$$x'(t) = F(x, y)$$
$$y'(t) = G(x, y)$$

Let $x(t) \equiv x_*, y(t) \equiv y_*$ be an equilibrium solution, i.e.

$$F(x_*, y_*) = 0$$
$$G(x_*, y_*) = 0 \ .$$

For solutions $[x(t), y(t)]^T$ to the original system, define the deviations from equilibrium $u(t), v(t)$ by

$$u(t) := x(t) - x_*$$
$$v(t) := y(t) - y_* \ .$$

Equivalently,

$$x(t) := x_* + u(t)$$
$$y(t) := y_* + v(t)$$

Thus

$$u' = x' = F(x, y) = F(x_* + u, y_* + v)$$
$$v' = y' = G(x, y) = G(x_* + u, y_* + v) \ .$$

Using partial derivatives, which measure rates of change in the coordinate directions, we can approximate

$$u' = F(x_* + u, y_* + v) = F(x_*, y_*) + \frac{\partial F}{\partial x}(x_*, y_*)\, u + \frac{\partial F}{\partial y}(x_*, y_*)\, v + \varepsilon_1\cancel{(u, v)}$$
$$v' = G(x_* + u, y_* + v) = G(x_*, y_*) + \frac{\partial G}{\partial x}(x_*, y_*)\, u + \frac{\partial G}{\partial y}(x_*, y_*)\, v + \varepsilon_2\cancel{(u, v)}$$

For differentiable functions, the error terms $\varepsilon_1, \varepsilon_2$ shrink more quickly than the linear terms, as $u, v \to 0$. Also, note that $F(x_*, y_*) = G(x_*, y_*) = 0$ because (x_*, y_*) is an equilibrium point. Thus the linearized system that approximates the non-linear system for $u(t), v(t)$, is (written in matrix vector form as):

$$\begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} \frac{\partial F}{\partial x}(x_*, y_*) & \frac{\partial F}{\partial y}(x_*, y_*) \\ \frac{\partial G}{\partial x}(x_*, y_*) & \frac{\partial G}{\partial y}(x_*, y_*) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}. \quad + \cancel{\begin{bmatrix} \varepsilon_1(u,v) \\ \varepsilon_2(u,v) \end{bmatrix}}$$

ignore

The matrix of partial derivatives is called the Jacobian matrix for the vector-valued function $[F(x, y), G(x, y)]^T$, evaluated at the point (x_*, y_*). Notice that it is evaluated at the equilibrium point.

People often use the subscript notation for partial derivatives to save writing, e.g F_x for $\frac{\partial F}{\partial x}$ and F_y for $\frac{\partial F}{\partial y}$.

Example 3) We will linearize the rabbit-squirrel (competition) model of the previous example, near the equilibrium solution $[4, 6]^T$. For convenience, here is that system:

$$x'(t) = 14x - 2x^2 - xy = F(x,y)$$
$$y'(t) = 16y - 2y^2 - xy = G(x,y)$$

3a) Use the Jacobian matrix method of linearizing they system at $[4, 6]^T$. In other words, as on the previous page, set

$$u(t) = x(t) - 4$$
$$v(t) = y(t) - 6$$

So, $u(t), v(t)$ are the deviations of $x(t), y(t)$ from 4, 6, respectively. Then use the Jacobian matrix computation to verify that the linearized system of differential equations that $u(t), v(t)$ approximately satisfy is

$$\begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} -8 & -4 \\ -6 & -12 \end{bmatrix} \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}.$$

$$J = \begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{bmatrix} = \begin{bmatrix} 14-4x-y & -x \\ -y & 16-4y-x \end{bmatrix}$$

$$J\begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} -8 & -4 \\ -6 & -12 \end{bmatrix}$$

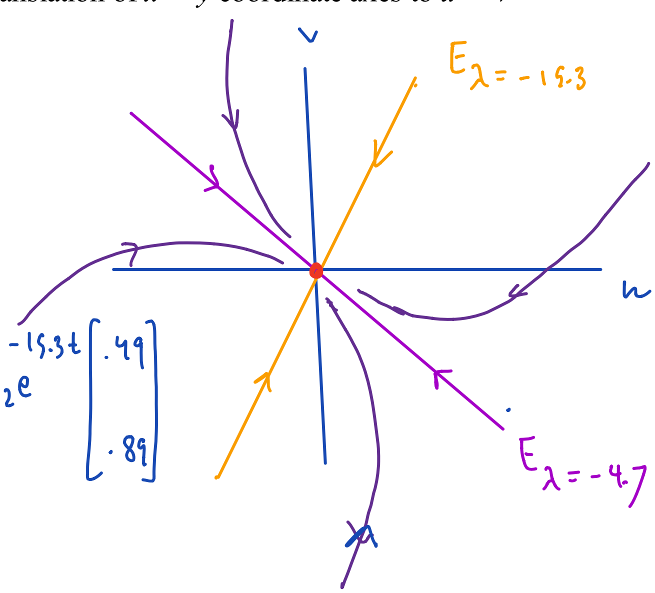
3b) The matrix in the linear system of DE's above has approximate eigendata:

$$\lambda_1 \approx -4.7, \quad \underline{v}_1 \approx [.79, -.64]^T$$
$$\lambda_2 \approx -15.3, \quad \underline{v}_2 \approx [.49, .89]^T$$

We can use the eigendata above to write down the general solution to the homogeneous (linearized) system, to make a rough sketch of the solution trajectories to the linearized problem near $[u, v]^T = [0, 0]^T$, and to classify the equilibrium solution using the Chapter 5 cases. Let's do that and then compare our work to the pplane output on the next page. As we'd expect, phase portrait for the linearized problem near $[u, v]^T = [0, 0]^T$ looks very much like the phase portrait for $[x, y]^T$ near $[4, 6]^T$. This is sensible, since the correspondence between (x, y) and (u, v) involves a translation of $x - y$ coordinate axes to $u - v$ coordinate axes, via the formula.

$$x = u + 4$$
$$y = v + 6$$

$$\begin{bmatrix} u \\ v \end{bmatrix} = c_1 e^{-4.7t} \begin{bmatrix} .79 \\ -.64 \end{bmatrix} + c_2 e^{-15.3t} \begin{bmatrix} .49 \\ .89 \end{bmatrix}$$



Monday : finish Chapter 6.

save time for 6.3.1 & 2 HW at end of class
(predatn - prey)

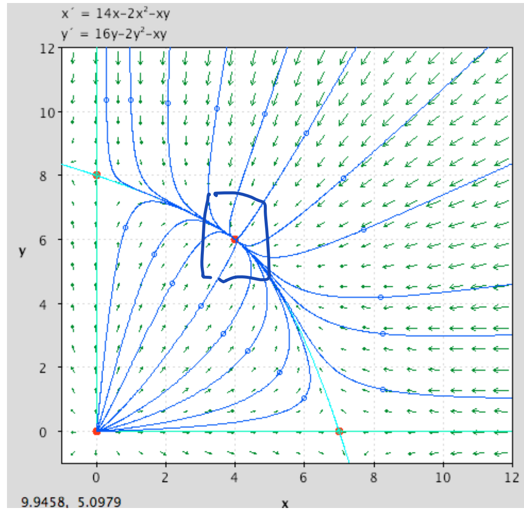
Tuesday : Course review notes ?

Wednesday (reading day) 8:05 - 9:25 go thru old final exam.

(Final exam Friday)

Linearization allows us to approximate and understand solutions to non-linear problems near equilbria:

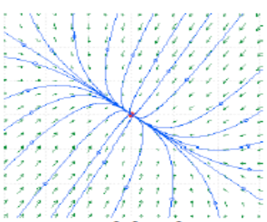
The non-linear problem and representative solution curves:



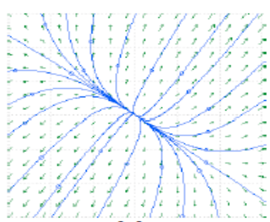
pplane will do the eigenvalue-eigenvector linearization computation for you, if you use the "find an equilibrium solution" option under the "solution" menu item.

```
Equilibrium Point:
    There is a nodal sink at (4, 6)
    Jacobian:
          -8        -4
          -6        -12
    The eigenvalues and eigenvectors are:
          -4.7085    (0.77218, -0.63541)
          -15.292    (0.48097, 0.87674)
```

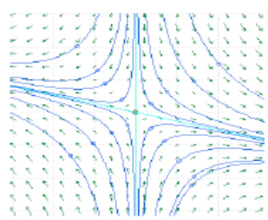
The solutions to the linearized system near $[u, v]^T = [0, 0]^T$ are close to the exact solutions for non-linear deviations, so under the translation of coordinates $u = x - x_*$, $v = y - y_*$ the phase portrait for the linearized system looks like the phase portrait for the non-linear system.



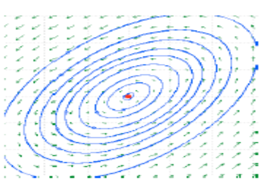
nodal sink
$\lambda_1, \lambda_2 < 0$



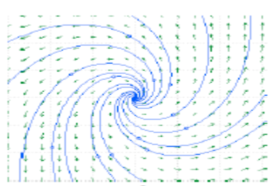
nodal source
$\lambda_1, \lambda_2 > 0$



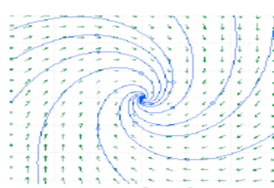
saddle point
$\lambda_1 < 0 < \lambda_2$



center
$\mathrm{Re}(\lambda)=0$



spiral source
$\mathrm{Re}(\lambda)>0$



spiral sink
$\mathrm{Re}(\lambda)<0$

Theorem: Let $[x_*, y_*]$ be an equilibrium point for a first order autonomous system of differential equations.
(i) If the linearized system of differential equations at $[x_*, y_*]$ has real eigendata, and either of an (asymptotically stable) nodal sink, an (unstable) nodal source, or an (unstable) saddle point, then the equilibrium solution for the non-linear system inherits the same stability and geometric properties as the linearized solutions.
(ii) If the linearized system has complex eigendata, and if $\Re(\lambda) \neq 0$, then the equilibrium solution for the non-linear system is also either an (unstable) spiral source or a (stable) spiral sink. If the linearization yields a (stable) center, then further work is needed to deduce stability properties for the nonlinear system.

borderline

Example 4 Returning to the non-linear pendulum

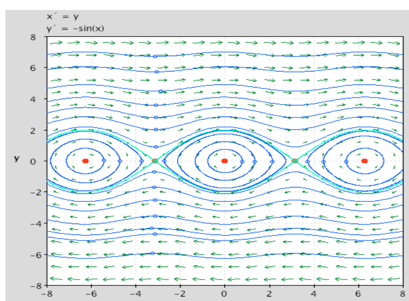
$$x'(t) = y \quad = F(x,y)$$
$$y'(t) = -\frac{g}{L}\sin(x). \quad = G(x,y)$$

$$\begin{bmatrix} x_* \\ y_* \end{bmatrix} = \begin{bmatrix} n\pi \\ 0 \end{bmatrix}$$

The solution trajectories ("orbits") follow level curves of the total energy function, which we repeat from page 1, recalling that $x(t) = \theta(t), y(t) = \theta'(t)$,

$$TE(t) = \frac{1}{2}m(L\,y)^2 + m\,g\,L(1 - \cos(x))$$

TE is strictly minimized at $\begin{bmatrix} n\pi \\ 0 \end{bmatrix}$ n even $\left(\begin{matrix} y=0 \\ \cos x = 1 \end{matrix} \right)$



If we compute the Jacobian matrix for this system, we get

$$J(x,y) = \begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L}\cos(x) & 0 \end{bmatrix}.$$

linear problem stable center. for non-linear borderline

- When $x = n\pi$ with n even (and $y = 0$),

$$J = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{bmatrix}$$

$$|J - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -\frac{g}{L} & -\lambda \end{vmatrix} = \lambda^2 + \frac{g}{L} = 0$$

roots $\lambda = \pm i\sqrt{\frac{g}{L}}$

the eigenvalues are $\lambda = \pm\, i\sqrt{\frac{g}{L}}$, so for the linearization we have a stable center, but this is the borderline case for the non-lin~~erar~~ear problem. Luckly these equilibrium points are exactly where the total energy function has its strict minimum value (of zero), and if a trajectory starts nearby its total energy is almost zero and the trajectory cannot wander away from the equilibrium point - so these are stable centers for the non-linear pendulum.

- When $x = n\pi$ with n odd (and $y = 0$),

$$J = \begin{bmatrix} 0 & 1 \\ +\frac{g}{L} & 0 \end{bmatrix}$$

$$|J - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ \frac{g}{L} & -\lambda \end{vmatrix} = \lambda^2 - \frac{g}{L} = 0$$

$$\lambda = \pm\sqrt{\frac{g}{L}}$$

the eigenvalues are $\lambda = \pm\sqrt{\frac{g}{L}}$, so for the linearization and the non-linear system we have an unstable saddle!

Saddle.

So for non-linear problem we have saddle pts at $\begin{bmatrix} n\pi \\ 0 \end{bmatrix}$ n odd.

Example 5) Consider the slightly damped pendulum with $\theta(t)$ with $\frac{g}{L} = 1$ and satisfying

$$\theta''(t) + .2\,\theta'(t) + \sin(\theta) = 0$$

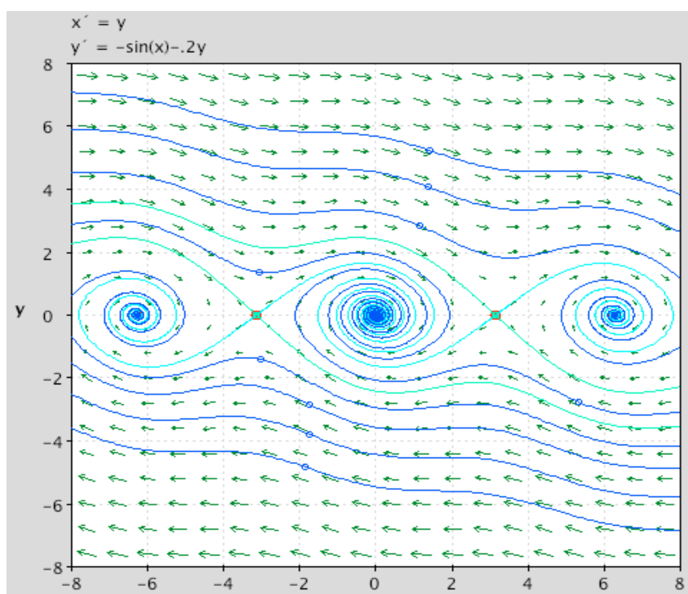
so that $[\theta(t), \theta'(t)]^T$ satisfies

$$x'(t) = y$$
$$y'(t) = -\sin(x) - 0.2\,y$$

$F(x,y) = y = 0$

$G(x,y) = -\sin x - .\cancel{2y} = 0$

One can check that we get the same equilibrium points as before, corresponding to the pendulum at rest vertically. The points $(x, y) = (n\pi, 0)$ with n odd are still saddles, but when n is even the stable centers are replaced with spiral sinks. This is an "underdamped" pendulum!



There are lots of interesting population models in section 9.2. Here's another competition model that looks deceptively like Example 2, except the competition got too intense (compare coefficients between the two systems).

<u>Example 6)</u>

$$x'(t) = 14x - x^2 - 2xy$$
$$y'(t) = 16y - y^2 - 2xy$$

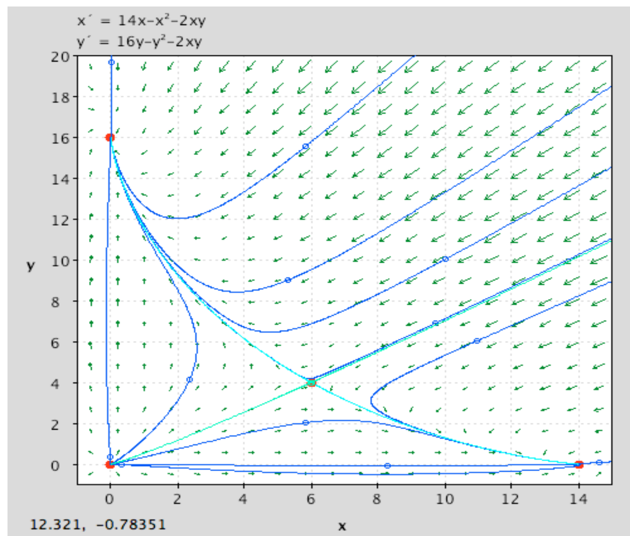
Do populations peacefully co-exist in this competition model? A little competition may be healthy, but too much maybe not so much. :-)

competition models

$x' = a_1 x - b_1 x^2 - c_1 xy$

$y' = a_2 x - b_2 x^2 - c_2 xy$

in earlier example, peaceful coexistence



Fact: (in this example $b_1 b_2 = 1$, $c_1 c_2 = 4$; in earlier example $b_1 b_2 = 4$, $c_1 c_2 = 1$)

if competition model has an equil. in 1st quadrant $\begin{bmatrix} x_* \\ y_* \end{bmatrix}$ with $x_*, y_* > 0$

& if $b_1 b_2 > c_1 c_2$ then $\begin{bmatrix} x_* \\ y_* \end{bmatrix}$ is stable & all $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \to \begin{bmatrix} x_* \\ y_* \end{bmatrix}$.

if $c_1 c_2 > b_1 b_2$ then $\begin{bmatrix} x_* \\ y_* \end{bmatrix}$ is unstable (if $x_0, y_0 > 0$) & one population dies out

Predator-Prey: $x(t)$ is the prey, $y(t)$ is the predator. One model:

$$x'(t) = ax - pxy = x(a - py)$$
$$y'(t) = -by + qxy = y(-b + qx)$$

natural region of interest is the first quadrant. Equilibrium solutions $(0, 0)$, $\left(\frac{b}{q}, \frac{a}{p}\right)$.

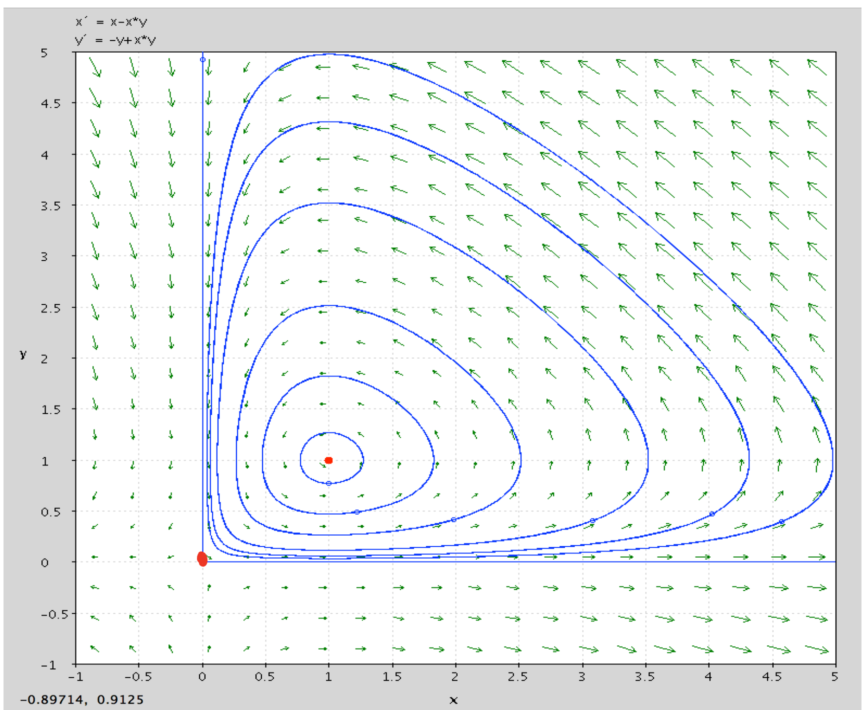
- Linearize at $\left(\frac{b}{q}, \frac{a}{p}\right)$ gives a stable center, need to do more work to deduce whether this equilibrium solution is a stable center for the non-linear system. (It is.)

Example:

$$x'(t) = x - xy = x(1 - y) = F(x,y)$$
$$y'(t) = -y + xy = y(-1 + x). = G(x,y)$$

$$J = \begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{bmatrix} = \begin{bmatrix} 1-y & -x \\ y & -1+x \end{bmatrix}$$

equil solns $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.



@ $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Saddle

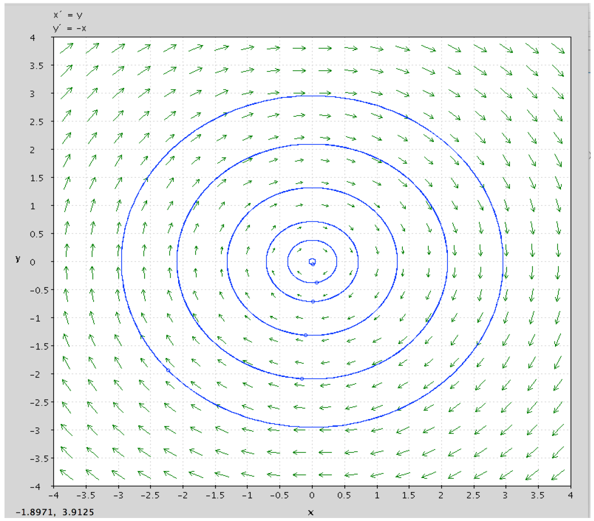
@ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$|J - \lambda I| = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$

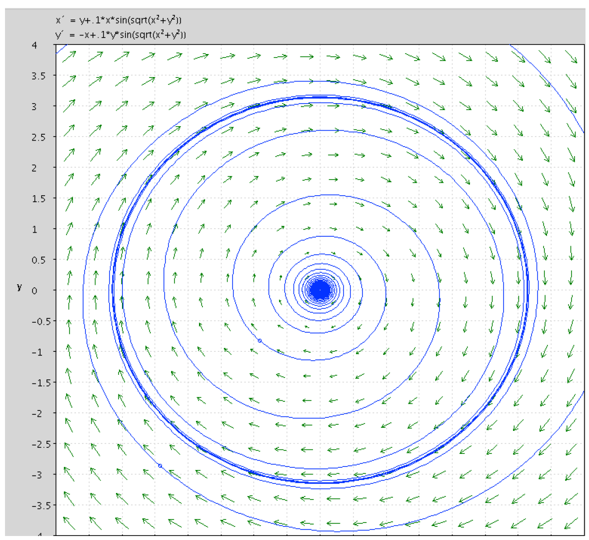
$\lambda = \pm i$

& stable center for linearization
borderline for non-linear,
but using separable DE's
you can show $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is stable

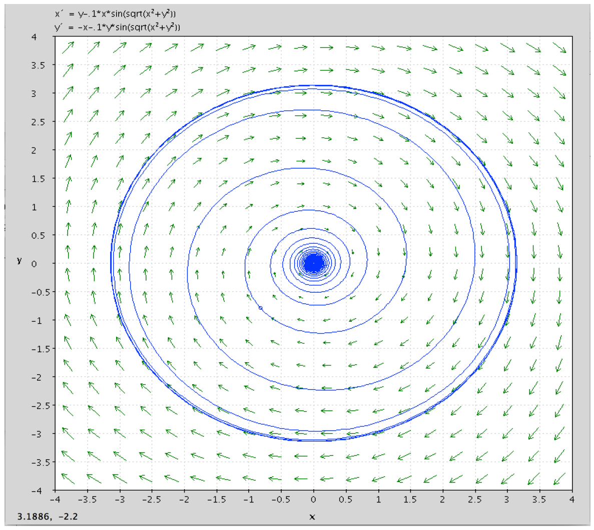
Stable center for linearization is borderline for nonlinear problem:



stable center



linearizes at $\begin{bmatrix}0\\0\end{bmatrix}$ to , but the higher order terms make the origin unstable, as if it was a spiral source



also linearizes at origin to stable center. In this case the higher order terms make origin behave like a spiral sink.