• Review course material in a new context
• 2270-2280 is a course in linear transformations

Fri Apr 21 and Mon Apr 24

Chapter 2: non-linear autonomous DE's

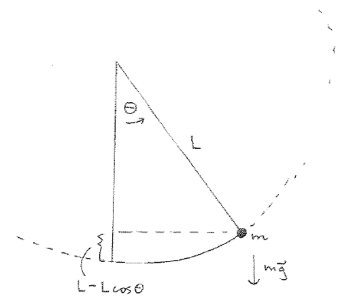
6.1-6.4 Nonlinear systems of autonomous first order differential equations and applications.

Introduction: In Chapter 2 we talked about equilibrium solutions to autonomous differential equations, i.e. constant solutions. Constant solutions are important because in real world dynamics the dynamics of a system are often only varying slightly from a constant values, especially if the constant configuration is stable. And whenever the situation is nearly in equilibrium, one can understand the dynamical system behavior by linearization. And, it turns out that the best way to understand equilibrium solutions is often to convert autonomous differential equations or systems to first order systems.

2nd order DE equiv to 1st order syst. of 2 DE's

Example 1) The rigid rod pendulum.

We've already considered a special case of this configuration, when the angle θ from vertical is near zero. Now assume that the pendulum is free to rotate through any angle $\theta \in \mathbb{R}$.



Earlier in the course we used conservation of energy to derive the dynamics for this (now) swinging, or possibly rotating, pendulum. There were no assumptions about the values of θ in that derivation of the non-linear DE (it was only when we linearized that we assumed θ was near zero). We began with the total energy

conservation of energy for eqtns of motion.

$$TE = KE + PE = \frac{1}{2}mv^2 + mgh$$

$$= \frac{1}{2}m(L\,\theta'(t))^2 + m\,g\,L\left(1 - \cos(\theta(t))\right)$$

And set $TE'(t) \equiv 0$ to arrive at the differential equation

$$\theta'(t)\left[\theta''(t) + \frac{g}{L}\sin(\theta(t))\right] = 0\,. \qquad (1)$$

We see that the constant solutions $\theta(t) = \theta_*$ must satisfy $\sin(\theta_*) = 0$, i.e. $\theta_* = n\,\pi$, $\pi \in \mathbb{R}$. In other words, the mass can be at rest at the lowest possible point (if θ is an even multiple of π), but also at the highest possible point (if θ is any odd multiple of π). We expect the lowest point configuration to be a "stable" constant solution, and the other one to be "unstable".

We will study these stability questions systematically using the equivalent first order system for

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \theta(t) \\ \theta'(t) \end{bmatrix} \qquad \begin{bmatrix} \theta \\ \theta' \end{bmatrix}' = \begin{bmatrix} \theta' \\ -\frac{g}{L}\sin\theta \end{bmatrix}$$

when $\theta(t)$ represents solutions the pendulum problem. You can quickly check that this is the system

$$\boxed{\begin{aligned} x'(t) &= y \\ y'(t) &= -\frac{g}{L}\sin(x) \end{aligned}}$$

const soltn

$0 = y_*$

$0 = -\frac{g}{L}\sin x_*$

Chtr 2

$P' = kP(M-P)$

constant soltns

$0 = kP(M-P)$

Notice that constant solutions of this system, $x' \equiv 0$, $y' \equiv 0$, equivalently

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_* \\ y_* \end{bmatrix} \text{ equals constant}$$

must satisfy $\boxed{y_* = 0}$, $\boxed{\sin(x_*) = 0}$. In other words, $x_* = n\,\pi$, $y_* = 0$ are the equilibrium solutions. These correspond to the constant solutions of the second order pendulum differential equation, $\theta = n\,\pi$, $\theta' = 0$.

Here's a phase portrait for the first order pendulum system, with $\frac{g}{L} = 1$, see below.

a) Locate the equilibrium points on the picture. ✓

b) Interpret the solution trajectories in terms of pendulum motion. ✓

c) Looking near each equilibrium point, and recalling our classifications of the origin for linear homogenous systems (spiral source, spiral sink, nodal source, nodal sink, saddle, stable center), how would you classify these equilbrium points and characterize their stability?

d) We'll talk about the general linearization procedure that explains the classifications in c, (and works for autonomous systems of more than two first order DEs, where there aren't such accessible pictures),after we do a populations example.

c) guess $(n\pi, 0)$ equil pts, n is even "stable center"

n is odd "saddle points"

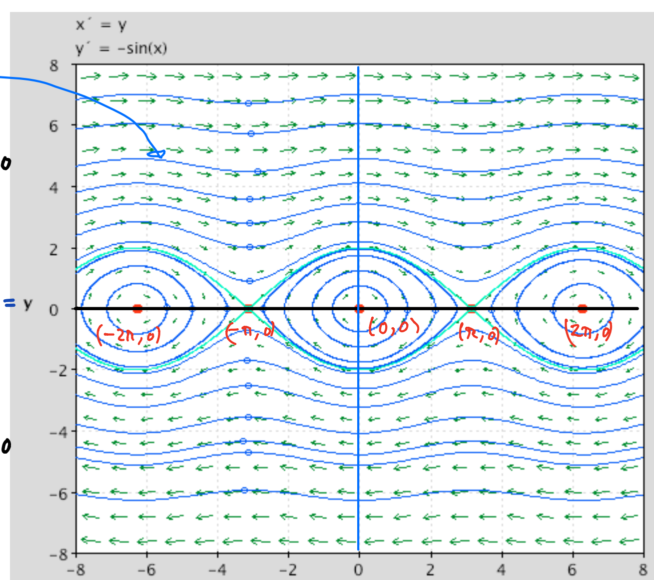
swing rotation w pos. θ'

$\theta' > 0$

$\theta' = y$

$\theta' < 0$

swing rot. w neg θ'



$\theta < 0$ $x = \theta$ $\theta > 0$

For reference, here are the precise definitions we're using:

The general (non-linear) system of two first order differential equations for $x(t), y(t)$ can be written as

$$x'(t) = F(x(t), y(t), t)$$
$$y'(t) = G(x(t), y(t), t)$$

which we often abbreviate, by writing

$$x' = F(x, y, t)$$
$$y' = G(x, y, t)\ .$$

If the rates of change F, G only depend on the values of $x(t), y(t)$ but not on t, i.e.

$$x' = F(x, y)$$
$$y' = G(x, y)$$

Chptr 2 $x' = f(x)$

then the system is called autonomous. Autonomous systems of first order DEs are the focus of Chapter 6, and are the generalization of one autonomous first order DE, as we studied in Chapter 2. In Chapter 6 the text restricts to systems of two equations as above, although most of the ideas generalize to more complicated autonomous systems with three or more interacting functions.

Constant solutions to an autonomous differential equation or system of DEs are called equilibrium solutions. Thus, equilibrium solutions $x(t) \equiv x_*, y(t) \equiv y_*$ have identically zero derivative and will correspond to solutions $[x_*, y_*]^T$ of the nonlinear algebraic system

also chptr 2.

$$F(x, y) = 0$$
$$G(x, y) = 0$$

$f(x) = 0$

- Equilibrium solutions $[x_*, y_*]^T$ to first order autonomous systems

$$x' = F(x, y)$$
$$y' = G(x, y)$$

are called stable if solutions to IVPs starting close (enough) to $[x_*, y_*]^T$ stay as close as desired.

- Equilibrium solutions are unstable if they are not stable.
- Equilibrium solutions $[x_*, y_*]^T$ are called asymptotically stable if they are stable and furthermore, IVP solutions that start close enough to $[x_*, y_*]^T$ converge to $[x_*, y_*]^T$ as $t \to \infty$.

(Notice these definitions are completely analogous to our discussion in Chapter 2.)

Same as Chapter 2

in 3210, 3220. $\forall \varepsilon > 0, \exists \delta > 0$ s.t. if $\left\| \begin{bmatrix} x_* \\ y_* \end{bmatrix} - \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \right\| < \delta$ then $\left\| \begin{bmatrix} x_* \\ y_* \end{bmatrix} - \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \right\| < \varepsilon$ $\forall t \geq 0$

stable.

unstable: $\exists \varepsilon > 0$ s.t. $\forall \delta > 0\ \exists \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ with $\left\| \begin{bmatrix} x_* \\ y_* \end{bmatrix} - \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \right\| < \delta$

but $\exists\ t_1 \geq 0$ with $\bigcirc \geq \varepsilon$

Example 2) Consider the "competing species" model from 9.2, shown below. For example and in appropriate units, $x(t)$ might be a squirrel population and $y(t)$ might be a rabbit population, competing on the same island sanctuary.

logistic ↓ ↓ competition

$$x'(t) = 14\,x - 2\,x^2 - x\,y$$
$$y'(t) = 16\,y - 2\,y^2 - x\,y\,.$$

Chptr. $P' = kP(M-P)$
$P' = aP - bP^2$

2a) Notice that if either population is missing, the other population satisfies a logistic DE. Discuss how the signs of third terms on the right sides of these DEs indicate that the populations are competing with each other (rather than, for example, acting in symbiosis, or so that one of them is a predator of the other).

Hint: to understand why this model is plausible for $x(t)$ consider the normalized ~~birth rate~~ fertility rate rate $\frac{x'(t)}{x(t)}$, as we did in Chapter 2.

$$\frac{x'(t)}{x(t)} = 14 - 2x - y$$

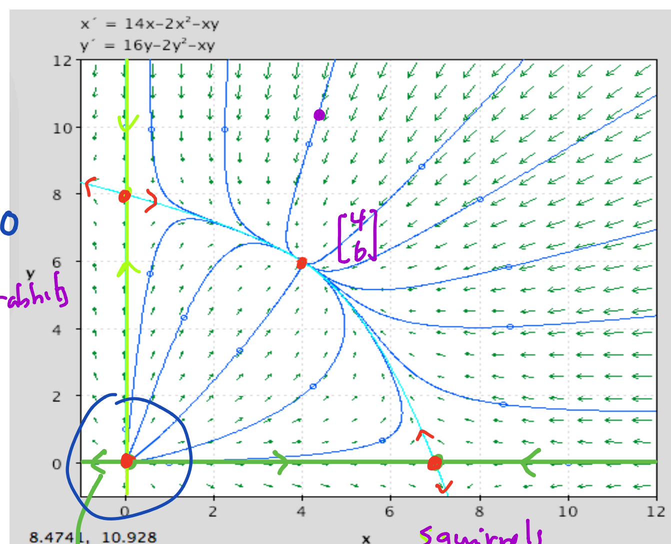
2b) Find the four equilibrium solutions to this competition model, algebraically.
2c) What does the phase portrait below indicate about the dynamics of this system?
2d) Based on our work in Chapter 5, how would you classify each of the four equilibrium points, including stability?

2b) $0 = 14x - 2x^2 - xy = x(14 - 2x - y)$
$0 = 16y - 2y^2 - xy = y(16 - 2y - x)$

$x = 0$: $y = 0$ → $\begin{bmatrix}0\\0\end{bmatrix}$; $16 - 2y - \cancel{x} = 0$ ($x = 0$), $y = 8$ → $\begin{bmatrix}0\\8\end{bmatrix}$

$14 - 2x - y = 0$: $y = 0$ → $\begin{bmatrix}7\\0\end{bmatrix}$; $16 - 2y - x = 0$ → $\begin{bmatrix}4\\6\end{bmatrix}$

4 equil. pts.



2c) if $x_0, y_0 > 0$ then $\lim_{t\to\infty} \begin{bmatrix}x(t)\\y(t)\end{bmatrix} = \begin{bmatrix}4\\6\end{bmatrix}$

if $y = 0$: $x' = 14x - 2x^2$
$x' = 2x(7 - x)$

Chptr 2

2d) $\begin{bmatrix}0\\0\end{bmatrix}$ (unstable) nodal source.

$\begin{bmatrix}7\\0\end{bmatrix}, \begin{bmatrix}0\\8\end{bmatrix}$ saddle (unstable)

$\begin{bmatrix}4\\6\end{bmatrix}$ (stable) nodal sink.

Linearization near equilibrium solutions is a recurring theme in differential equations and in this Math 2280 course. (You may have forgotten, but the "linear drag" velocity model, Newton's law of cooling, and the damped spring equation were all linearizations!!) It's important to understand how to linearize in general, because the linearized differential equations can often be used to understand stability and solution behavior near the equilibrium point, for the original differential equations. Today we'll talk about linearizing systems systems of DE's, which we've not done before in this class.

An easy case of linearization in Example 2 is near the equilbrium solution $[x_*, y_*]^T = [0, 0]^T$. It's pretty clear that our population system

$$x'(t) = 14\,x - \cancel{2\,x^2} - \cancel{x\,y}$$
$$y'(t) = 16\,y - \cancel{2\,y^2} - \cancel{x\,y}$$

linearizes to

$$x'(t) = 14\,x$$
$$y'(t) = 16\,y$$

i.e.

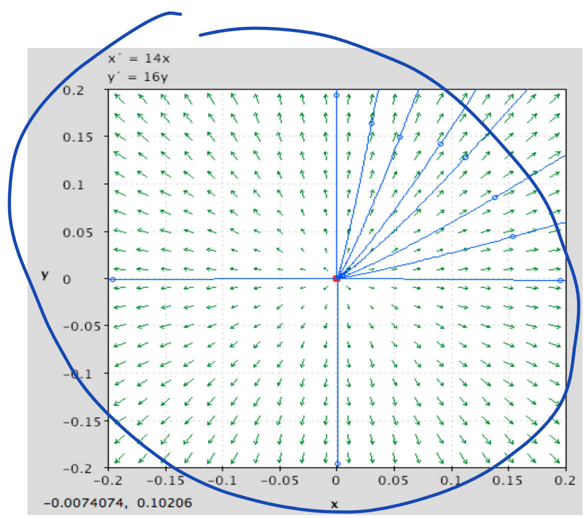
$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 14 & 0 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

The eigenvalues are the diagonal entries, and the eigenvectors are the standard basis vectors, so

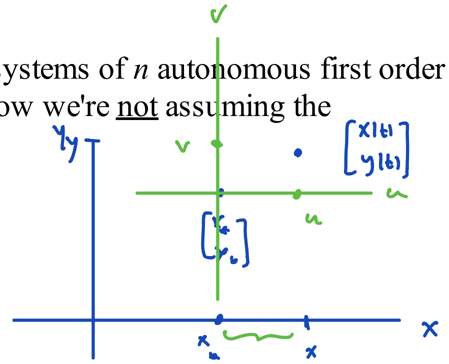
$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{14\,t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{16\,t} \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

Notice how the phase portrait for the linearized system looks like that for the non-linear system, near the origin:

looks like ↑



How to linearize with multivariable Calculus: (This would work for systems of n autonomous first order differential equations, but we focus on $n = 2$ in this chapter. Notice how we're not assuming the equilbrium point is the origin. Here's the general system:



$$x'(t) = F(x, y)$$
$$y'(t) = G(x, y)$$

Let $x(t) \equiv x_*, y(t) \equiv y_*$ be an equilibrium solution, i.e.

$$F(x_*, y_*) = 0$$
$$G(x_*, y_*) = 0 \ .$$

For solutions $[x(t), y(t)]^T$ to the original system, define the deviations from equilibrium $u(t), v(t)$ by

$$\left.\begin{aligned} u(t) &:= x(t) - x_* \\ v(t) &:= y(t) - y_* \ . \end{aligned}\right\}$$

Equivalently,

$$\left.\begin{aligned} x(t) &:= x_* + u(t) \\ y(t) &:= y_* + v(t) \end{aligned}\right\}$$

Thus

$$u' = x' = F(x, y) = F(x_* + u, y_* + v)$$
$$v' = y' = G(x, y) = G(x_* + u, y_* + v) \ .$$

Using partial derivatives, which measure rates of change in the coordinate directions, we can approximate

$$u' = F(x_* + u, y_* + v) = F(x_*, y_*) + \frac{\partial F}{\partial x}(x_*, y_*)\, u + \frac{\partial F}{\partial y}(x_*, y_*)\, v + \varepsilon_1\cancel{(u, v)}$$
$$v' = G(x_* + u, y_* + v) = G(x_*, y_*) + \frac{\partial G}{\partial x}(x_*, y_*)\, u + \frac{\partial G}{\partial y}(x_*, y_*)\, v + \varepsilon_2\cancel{(u, v)}$$

For differentiable functions, the error terms $\varepsilon_1, \varepsilon_2$ shrink more quickly than the linear terms, as $u, v \to 0$. Also, note that $F(x_*, y_*) = G(x_*, y_*) = 0$ because (x_*, y_*) is an equilibrium point. Thus the linearized system that approximates the non-linear system for $u(t), v(t)$, is (written in matrix vector form as):

$$\begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} \frac{\partial F}{\partial x}(x_*, y_*) & \frac{\partial F}{\partial y}(x_*, y_*) \\ \frac{\partial G}{\partial x}(x_*, y_*) & \frac{\partial G}{\partial y}(x_*, y_*) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} . \quad + \cancel{\begin{bmatrix} \varepsilon_1(u,v) \\ \varepsilon_2(u,v) \end{bmatrix}}$$

ignore

The matrix of partial derivatives is called the Jacobian matrix for the vector-valued function $[F(x, y), G(x, y)]^T$, evaluated at the point (x_*, y_*). Notice that it is evaluated at the equilibrium point.

People often use the subscript notation for partial derivatives to save writing, e.g F_x for $\frac{\partial F}{\partial x}$ and F_y for $\frac{\partial F}{\partial y}$.

Example 3) We will linearize the rabbit-squirrel (competition) model of the previous example, near the equilibrium solution $[4, 6]^T$. For convenience, here is that system:

$$x'(t) = 14x - 2x^2 - xy = F(x,y)$$
$$y'(t) = 16y - 2y^2 - xy = G(x,y)$$

3a) Use the Jacobian matrix method of linearizing they system at $[4, 6]^T$. In other words, as on the previous page, set

$$u(t) = x(t) - 4$$
$$v(t) = y(t) - 6$$

So, $u(t), v(t)$ are the deviations of $x(t), y(t)$ from 4, 6, respectively. Then use the Jacobian matrix computation to verify that the linearized system of differential equations that $u(t), v(t)$ approximately satisfy is

$$\begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} -8 & -4 \\ -6 & -12 \end{bmatrix} \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}.$$

$$J = \begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{bmatrix} = \begin{bmatrix} 14 - 4x - y & -x \\ -y & 16 - 4y - x \end{bmatrix}$$

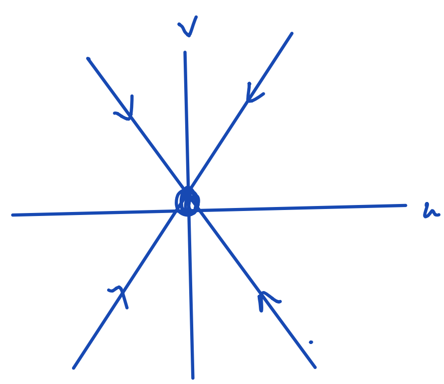
$$J\begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} -8 & -4 \\ -6 & -12 \end{bmatrix}$$

3b) The matrix in the linear system of DE's above has approximate eigendata:

$$\lambda_1 \approx -4.7, \quad \underline{v}_1 \approx [.79, -.64]^T$$
$$\lambda_2 \approx -15.3, \quad \underline{v}_2 \approx [.49, .89]^T$$

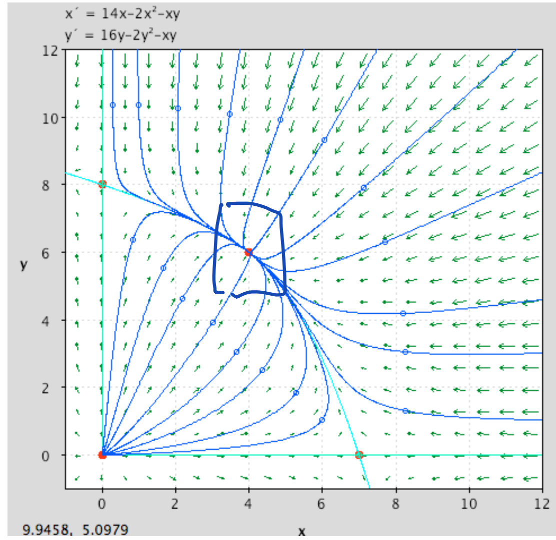
We can use the eigendata above to write down the general solution to the homogeneous (linearized) system, to make a rough sketch of the solution trajectories to the linearized problem near $[u, v]^T = [0, 0]^T$, and to classify the equilibrium solution using the Chapter 5 cases. Let's do that and then compare our work to the pplane output on the next page. As we'd expect, phase portrait for the linearized problem near $[u, v]^T = [0, 0]^T$ looks very much like the phase portrait for $[x, y]^T$ near $[4, 6]^T$. This is sensible, since the correspondence between (x, y) and (u, v) involves a translation of $x - y$ coordinate axes to $u - v$ coordinate axes, via the formula.

$$x = u + 4$$
$$y = v + 6$$



Linearization allows us to approximate and understand solutions to non-linear problems near equilbria:

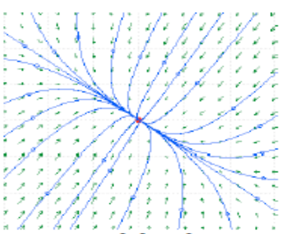
The non-linear problem and representative solution curves:



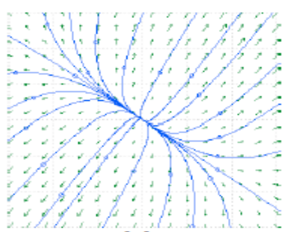
pplane will do the eigenvalue-eigenvector linearization computation for you, if you use the "find an equilibrium solution" option under the "solution" menu item.

```
Equilibrium Point:
    There is a nodal sink at (4, 6)
    Jacobian:
            -8          -4
            -6          -12
    The eigenvalues and eigenvectors are:
            -4.7085    (0.77218, -0.63541)
            -15.292    (0.48097, 0.87674)
```

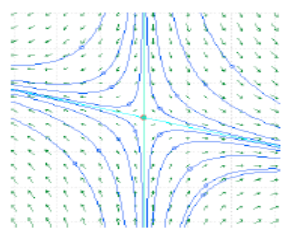
The solutions to the linearized system near $[u, v]^T = [0, 0]^T$ are close to the exact solutions for non-linear deviations, so under the translation of coordinates $u = x - x_*$, $v = y - y_*$ the phase portrait for the linearized system looks like the phase portrait for the non-linear system.



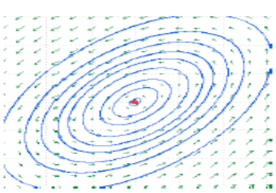
nodal sink
$\lambda_1, \lambda_2 < 0$



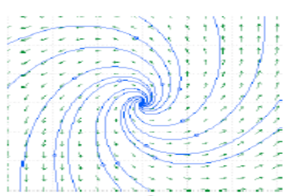
nodal source
$\lambda_1, \lambda_2 > 0$



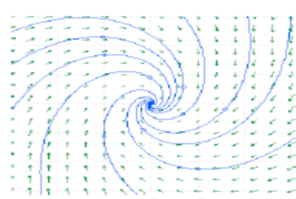
saddle point
$\lambda_1 < 0 < \lambda_2$



center
$\mathrm{Re}(\lambda)=0$



spiral source
$\mathrm{Re}(\lambda)>0$



spiral sink
$\mathrm{Re}(\lambda)<0$