We talked about differentiating Fourier series term by term in Friday's notes. There is also:

<u>Theorem</u> If f is piece-wise continuous, $2\,L$-periodic, with Fourier series

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n\frac{\pi}{L}t\right) + \sum_{n=1}^{\infty} b_n \sin\left(n\frac{\pi}{L}t\right)$$

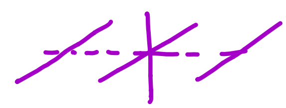
$\int \cos \frac{n\pi}{L} t = \frac{L}{n\pi} \sin\frac{n\pi}{L} t$

then the antiderivative may be computed by term by term antidifferentiation, and the corresponding series will converge for each t:

$$\int_0^t f(s)\,\mathrm{d}s = \frac{a_0}{2}t + \sum_{n=1}^{\infty} a_n \left(\frac{L}{n\pi}\right)\sin\left(n\frac{\pi}{L}t\right) + \sum_{n=1}^{\infty} b_n\left(\frac{L}{n\pi}\right)\left(-\cos\left(n\frac{\pi}{L}t\right)+1\right).$$

<u>Exercise 3)</u> (This is the first part of your homework exercise 9.3.19)
Start with

"t"



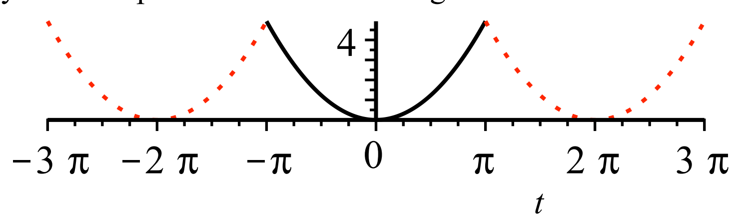
$\int_0^t \, dt$

$$t = saw(t) \sim 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin(n\,t)$$

and integrate to get the $2\,\pi$- periodic function that on $[-\pi, \pi]$ is given by

$$\boxed{\frac{t^2}{2} = \frac{\pi^2}{6} + 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos(n\,t).}$$

Hint: The value of the constant term is easiest to compute as $\frac{a_0}{2}$. If you compare to the definite integral formula in the Theorem you will reproduce one of the "magic" series.



In your homework you will antidifferentiate twice more to get a formula for the periodic extension of $g(t) = \frac{t^4}{24}$ and some more magic formulas.

$$\int_0^t saw(\tau)\,d\tau = 2\int_0^t \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\tau \; d\tau$$

$-\pi \le t \le \pi$:

$$\frac{t^2}{2} = 2\sum_{n=1}^{\infty} \underbrace{\int_0^t \frac{(-1)^{n+1}}{n}\sin n\tau\,d\tau}_{\frac{(-1)^{n+1}}{n}\left[-\frac{\cos n\tau}{n}\right]_0^t}$$

$$= 2\sum_{n=1}^{\infty}(-1)^n\left(\frac{\cos nt}{n^2} - \frac{1}{n^2}\right)$$

$$\boxed{\frac{t^2}{2} = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nt}$$

"magic sum" $1 - \frac{1}{4} + \frac{1}{9} - \ldots$

magic formulas so far:

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6} = \zeta(2)$$

$$\sum_{n \text{ odd}}\frac{1}{n^2} = \frac{\pi^2}{8}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

wikipedia.

Riemann-Zeta fcn. $\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$, $s \in \mathbb{C}$
connects to prime # theorem

also know $\frac{t^2}{2} = \frac{a_0}{2} + \sum a_n \cos nt + \underbrace{\sum b_n \sin nt}_{0.}$

alternate

@ $t = \pi$

$$\frac{\pi^2}{2} = 2S + 2\underbrace{\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}(-1)^n}_{\sum_{n=1}^{\infty} \frac{1}{n^2}}$$

$$\frac{\pi^2}{2} = 2S + 2\frac{\pi^2}{6}$$

$$\frac{\pi^2}{6} = 2S$$

$$\frac{\pi^2}{12} = S.$$

so $2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{a_0}{2}$ ✓

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{t^2}{2}\,dt \quad ✓$$

oops, our $f(t)$ is $\frac{t^2}{2}$, not t^2

$$= \frac{1}{\pi}\int_0^{\pi} \frac{t^2}{2}\,dt \quad ✓$$

$$= \frac{1}{\pi}\frac{\pi^3}{\cancel{3}\,6} \quad ✓$$

$$= \frac{1}{\cancel{3}\,6}\pi^2 \quad ✓$$

$$\frac{a_0}{2} = 2S = \frac{\pi^2}{6}$$

$$S = \frac{\pi^2}{12} \quad ✓$$

$$\boxed{\text{so} \quad \frac{t^2}{2} = \frac{\pi^2}{6} + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nt} \quad -\pi < t < \pi$$

integrate two more times from 0 to t in HW 9.3.19

magic formulas for $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$

$$\sum_{n \text{ odd}} \frac{1}{n^4} = \frac{\pi^4}{96}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} = \frac{7\pi^4}{720}$$

• today's notes like lab HW.
• magic formulas in Monday notes continued ... these are like book HW.

Math 2280-001
Wed Apr 19

- Finish Monday's notes first. Then ...

9.4 Forced oscillation problems via Fourier Series. Today we will revisit the forced oscillation problems of last ~~Friday~~ Wed., where we predicted whether or not resonance would occur, and then tested our predictions with the convolution solutions. Using Fourier series expansions for the forcing function one can say precisely whether or not there will be resonance. We will be studying the differential equations

$$x''(t) + c\,x'(t) + \omega_0^2 x(t) = f(t)$$

for various forcing functions $f(t)$. (We have divided the original mass-spring DE by the mass m and relabeled the damping coefficient and forcing functions.) For most of the lecture we consider undamped configurations, $c = 0$.

Warm-up Exercise 1) (This was the final exercise last Wednesday, when we were using convolution integrals to study forced oscillation problems.)
Use superposition to find particular solutions, and discuss whether or not resonance will occur in the following two forced oscillation problems. Notice that the period of the forcing function is 6π in a (not the natural period). In b the period of the forcing function is 2π. And yet, the resonance occurs in a, and not in b.

$\frac{2\pi}{w_0} = 2\pi = T_0$ period $= \text{lcm}(2\pi, 6\pi) = 6\pi$

a)

$$x''(t) + x(t) = \cos(t) + \sin\left(\frac{t}{3}\right).$$

did! Resonance.

b)

$$x''(t) + x(t) = \underbrace{\cos(2t) - 3\sin(3t) + \cos(6t)}.$$

(not in lab).

Hint: There's a table of particular solutions at the end of today's notes.

Period $= \text{lcm}\left(\pi, \frac{2\pi}{3}, \frac{\pi}{3}\right) = 2\pi = T_0$
but no resonance

Exercise 1 is indicative of how we can understand resonance phenomena for forced oscillation problems with general periodic forcing functions f:

$$x''(t) + \omega_0^2\, x(t) = f(t),$$

where $f(t)$ has period $P = 2\,L$. Compute the Fourier series for f:

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n\frac{\pi}{L}t\right) + \sum_{n=1}^{\infty} b_n \sin\left(n\frac{\pi}{L}t\right)$$

with

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\,\mathrm{d}t \qquad \left(\text{so } \frac{a_0}{2} = \frac{1}{2\,L}\int_{-L}^{L} f(t)\,\mathrm{d}t \text{ is the average value of } f\right)$$

$$a_n := \left\langle f, \cos\left(n\frac{\pi}{L}\,t\right)\right\rangle = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(n\frac{\pi}{L}\,t\right)\,\mathrm{d}t,\ \ n \in \mathbb{N}$$

$$b_n := \left\langle f, \sin\left(n\frac{\pi}{L}\,t\right)\right\rangle = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(n\frac{\pi}{L}\,t\right)\,\mathrm{d}u,\ n \in \mathbb{N}$$

As long as no (non-zero) term in the Fourier series has an angular frequency of ω_0, there will be no resonance. In fact, in this case the infinite sum of (undetermined coefficients) particular solutions will converge to a bounded particular solution. For sure there will NOT be resonance if it's true for all $n \in \mathbb{N}$ that

$$\omega_n := n\frac{\pi}{L} \neq \omega_0$$

but even if some ω_n does equal ω_0 there won't be resonance unless either a_n or b_n is nonzero.

Conversely, if the Fourier series of f does contain $\cos(\omega_0 t)$ or $\sin(\omega_0 t)$ terms, those terms will cause resonance.

Recall the first "resonance game" example from last Friday:

$$x''(t) + x(t) = square(t)$$

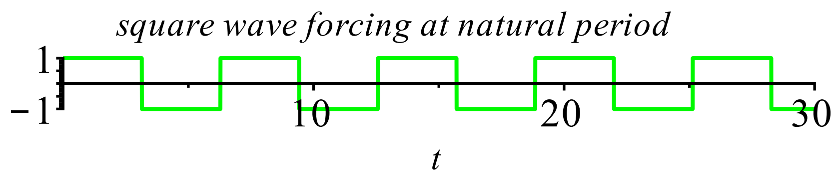
with

$$square(t) = \begin{cases} -1 & -\pi < t < 0 \\ 1 & 0 < t < \pi \end{cases}$$

and 2π-periodic. This forcing function appeared to cause resonance:
Here's a formula for $square(t)$ valid for $0 \le t \le 11\pi$, and last Friday's results:

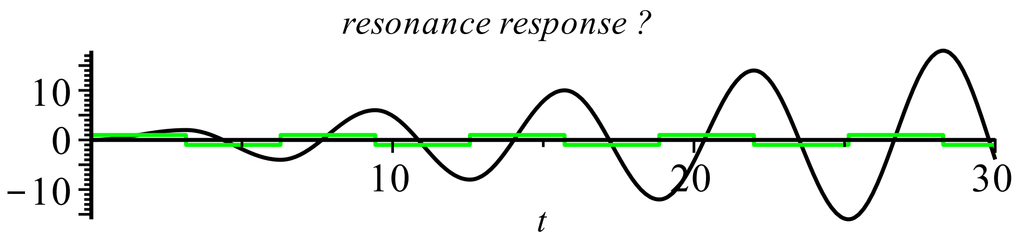
> $with(plots):$

> $square := t \to 1 + 2 \cdot \left(\sum_{n=1}^{10} (-1)^n \cdot \text{Heaviside}(t - n \cdot \text{Pi}) \right):$
> $plot1a := plot(square(t), t = 0..30, color = green):$
> $display(plot1a, title = \text{`square wave forcing at natural period`});$



Convolution solution formula and graph:

> $x1 := t \to \int_0^t \sin(\tau) \cdot square(t - \tau)\, d\tau :$
> $plot1b := plot(x1(t), t = 0..30, color = black):$
> $display(\{plot1a, plot1b\}, title = \text{`resonance response ?`});$



Exercise 2 Use the Fourier series for $square(t)$ that we've found before

$$square(t) = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \sin(n\,t)$$

and infinite superposition to find a particular solution to

$$x''(t) + x(t) = square(t)$$

that explains why resonance occurs. Make use of the undetermined coefficients particular solution formulas at the end of today's notes.

$$x'' + x = \frac{4}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \cdots \right)$$

$$x = x_p + x_H$$

$$x = c_1 \cos t + c_2 \sin t + \frac{4}{\pi}\left(-\frac{t}{2}\cos t + \frac{1}{3}\frac{1}{1-9}\sin 3t + \cdots \right)$$

$$= c_1 \cos t + c_2 \sin t - \frac{2}{\pi} t \cos t + \sum_{\substack{n \text{ odd} \\ n \ge 3}} \underbrace{\left(\frac{1}{n}\frac{1}{1-n^2}\right)}\sin nt$$

resonance term.

$$\left|\sum\right| \le \sum|\ |$$

$$\sum_{\substack{n \text{ odd} \\ n \ge 3}} \left|\ \ \right| \le \sum_{\substack{n \text{ odd} \\ n \ge 3}} \left|\frac{1}{n-n^3}\right| \cdot 1 = \sum_{\substack{n \text{ odd} \\ n \ge 3}} \frac{1}{n^3-n} \le \sum_{\substack{n \text{ odd} \\ n \ge 3}} \frac{1}{n^2}$$

$$= \frac{\pi^2}{8} - 1 < .2$$

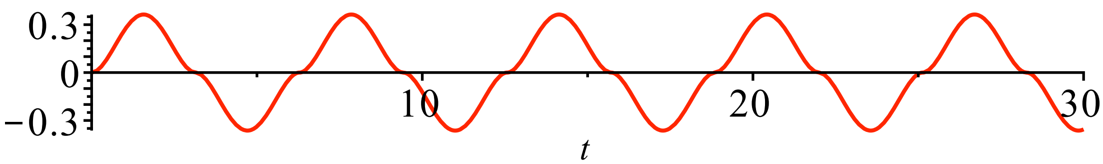
term that caused resonance.

(for IVP $x(0)=0$, $x'(0)=0$, $c_1=0$, $c_2 \neq 0$)

If we remove the $\sin(t)$ term from the square wave forcing function, and re-use the convolution formula, we see that we've eliminated the resonance:

$$> x2 := t \rightarrow \int_0^t \sin(\tau)\cdot\left(square(t-\tau) - \underbrace{\frac{4}{\pi}\cdot\sin(t-\tau)}\right) d\tau : \quad = \int_0^t w(\tau)\, f(t-\tau)\, d\tau$$

plot(x2(t), t = 0 ..30);



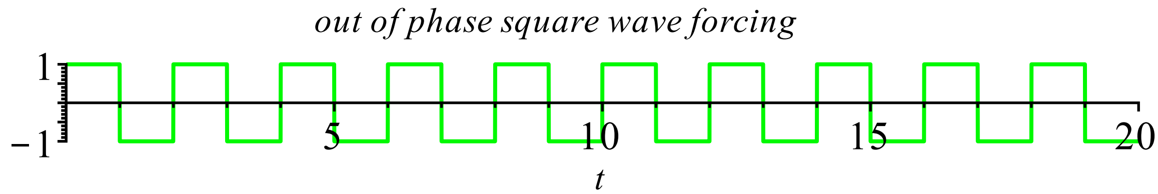
Exercise 2) Understand Example 3 from last Friday, using Fourier series:

$$x''(t) + x(t) = f_3(t)$$

Example 3) Forcing not at the natural period, e.g. with a square wave having period $T = 2$.

$$> f3 := t \to 1 + 2 \cdot \sum_{n=1}^{20} (-1)^n \cdot \text{Heaviside}(t - n) :$$

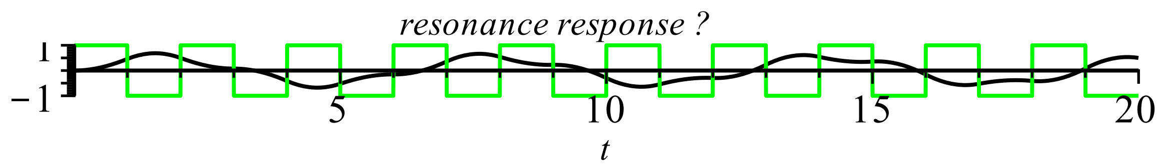
$plot3a := plot(f3(t), t = 0 ..20, color = green) :$
$display(plot3a, title = \text{`out of phase square wave forcing`});$

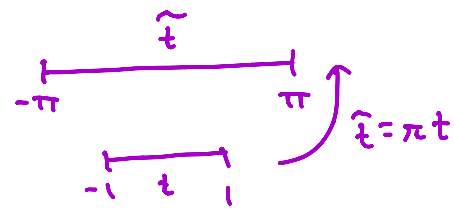


This forcing function did not cause resonance:

$$> x3 := t \to \int_0^t \sin(\tau) \cdot f3(t - \tau) \, d\tau :$$

$plot3b := plot(x3(t), t = 0 ..20, color = black) :$
$display(\{plot3a, plot3b\}, title = \text{`resonance response ?`});$





Hint: By rescaling we can express $f_3(t) = square(\pi t) = \frac{4}{\pi} \sum_{n \, odd} \frac{1}{n} \sin(n \pi t)$.

$$x'' + x = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \sin(n\pi t)$$

$x = x_H + x_p$

$$x(t) = c_1 \cos t + c_2 \sin t + \underbrace{\frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \left(\frac{1}{1 - n^2\pi^2} \right) \sin(n\pi t)}$$

$\omega_0 = 1$

$\omega = \omega_n = n\pi$

$$| \quad | \le \frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{n^3\pi^2 - n}$$

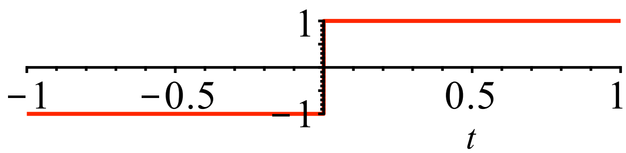
$$n^3\pi^2 - n \ge n^2\pi^2$$

$$\le \frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{n^2\pi^2} = \frac{4}{\pi^3} \left(\frac{\pi^2}{8} \right) < \frac{1}{2\pi}$$

Brute force tech check of Fourier coefficients in previous example:

> $f := t \rightarrow -1 + 2 \cdot \mathrm{Heaviside}(t);$
> $plot(f(t), t = -1 .. 1);$
> $L := 1;$

$$f := t \rightarrow -1 + 2\,\mathrm{Heaviside}(t)$$



$$L := 1 \qquad \textbf{(1)}$$

> $a0 := \frac{1}{L} \cdot \int_{-L}^{L} f(t)\, \mathrm{d}t;$
> $assume(n, integer);$ # *this will let Maple attempt to evaluate the integrals*
> $a := n \rightarrow \frac{1}{L} \cdot \int_{-L}^{L} f(t) \cdot \cos\left(\frac{n \cdot \pi}{L} t\right) \mathrm{d}t :$
> $b := n \rightarrow \frac{1}{L} \cdot \int_{-L}^{L} f(t) \cdot \sin\left(\frac{n \cdot \pi}{L} t\right) \mathrm{d}t :$
> $a(n);$
> $b(n);$

$$a0 := 0$$

$$0$$

$$\frac{-(-1)^{n\sim} + 2}{n\sim\, \pi} - \frac{(-1)^{n\sim}}{n\sim\, \pi} \qquad \textbf{(2)}$$

>

>

Practical resonance example:

Exercise 3 The steady periodic solution to the differential equation

$$x''(t) + .2\cdot x'(t) + 1\,x(t) = square(t) = \frac{4}{\pi}\sum_{n\ odd}\frac{1}{n}\sin(nt)$$

exhibits practical resonance. Explain this with Fourier series. Hint: Use

$$square(t) = \frac{4}{\pi}\sum_{n\ odd}\frac{1}{n}\sin(n\,t)$$

$$= \frac{4}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \cdots\right)$$

the table of particular solutions at the end of today's notes.

$$x = x_p + x_H \quad (x_H \xrightarrow{t\to\infty} 0)$$

$$x(t) = x_{sp}(t) + x_{tr}(t)$$

$c = .2$

$\omega_0 = 1$

$\omega_n = n$

$\omega_1 = \omega_0 = 1$

$$x_p = \frac{4}{\pi}\left(\left(\frac{1}{\sqrt{0 + .04}}\sin(t-\alpha_1) + \sum_{\substack{n\ odd\\ n\geq 3}}\frac{1}{n}\left(\frac{1}{\sqrt{(n^2-1)^2 + .04n^2}}\right)\sin(nt-\alpha_n)\right)\right)$$

α_1, α_n from table.

$$x_p = \frac{20}{\pi}\sin(t-\alpha_1) + \frac{4}{\pi}\cdot \bigcirc$$

$$\left|\frac{4}{\pi}\cdot\bigcirc\right| \leq \underbrace{\frac{4}{\pi}\sum_{\substack{n\ odd\\ n\geq 3}}\frac{1}{n(n^2-1)}}_{\text{tiny.}}$$

Particular solutions from Chapter 3 or Laplace transform table:

$$x''(t) + \omega_0^2\, x(t) = A\, sin(\omega t)$$

$$x_P(t) = \frac{A}{\omega_0^2 - \omega^2} \sin(\omega t) \quad \text{when } \omega \neq \omega_0$$

$$x_P(t) = -\frac{t}{2\,\omega_0}\, A \cos(\omega_0\, t) \quad \text{when } \omega = \omega_0$$

....

$$x''(t) + \omega_0^2\, x(t) = A \cos(\omega t)$$

$$x_P(t) = \frac{A}{\omega_0^2 - \omega^2} \cos(\omega t) \quad \text{when } \omega \neq \omega_0$$

$$x_P(t) = \frac{t}{2\,\omega_0}\, A \sin(\omega_0\, t) \quad \text{when } \omega = \omega_0$$

....

$$x'' + c\,x' + \omega_0^2\, x = A \cos(\omega t) \qquad c > 0$$

$$x_P(t) = x_{sp}(t) = C \cos(\omega t - \alpha)$$

with

$$C = \frac{A}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + c^2\omega^2}} \ .$$

$$\cos(\alpha) = \frac{\omega_0^2 - \omega^2}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + c^2\omega^2}}$$

$$\sin(\alpha) = \frac{c\,\omega}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + c^2\omega^2}} \ .$$

....

$$x'' + c\,x' + \omega_0^2\, x = A \sin(\omega t) \qquad c > 0$$

$$x_P(t) = x_{sp}(t) = C \sin(\omega t - \alpha)$$

with

$$C = \frac{A}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + c^2\omega^2}} \ .$$

$$\cos(\alpha) = \frac{\omega^2 - \omega_0^2}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + c^2\omega^2}}$$

$$\sin(\alpha) = \frac{c\,\omega}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + c^2\omega^2}}$$