Convergence Theorems (These require some careful mathematical analysis to prove - they are often discussed in Math 5210, for example.)

Theorem 1 Let $f: \mathbb{R} \to \mathbb{R}$ be 2π-periodic and piecewise continuous. Let

$$f_N = proj_{V_N} f = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(nt) + \sum_{n=1}^{N} b_n \sin(nt)$$

be the Fourier series truncated at N. Then

$$\lim_{n \to \infty} \| f - f_N \| = \lim_{n \to \infty} \left[\frac{1}{\pi} \int_{-\pi}^{\pi} \left(f(t) - f_N(t) \right)^2 \, dt \right]^{\frac{1}{2}} = 0.$$

In other words, the distance between f_N and f converges to zero, where we are using the distance function that we get from the inner product,

$$dist(f, g) = \| f - g \| = \sqrt{\langle f - g, f - g \rangle} = \left[\frac{1}{\pi} \int_{-\pi}^{\pi} \left(f(t) - g(t) \right)^2 \, dt \right]^{\frac{1}{2}}.$$

Theorem 2 If f is as in Theorem 1, and is (also) piecewise differentiable with at most jump discontinuities, then

(i) for any t_0 such that f is differentiable at t_0

$$\lim_{N \to \infty} f_N(t_0) = f(t_0) \quad \text{(pointwise convergence).}$$

(ii) for any t_0 where f is not differentiable (but is either continuous or has a jump discontinuity), then

$$\lim_{N \to \infty} f_N(t_0) = \frac{1}{2} \left(f_-(t_0) + f_+(t_0) \right)$$

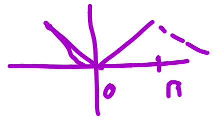
where

$$f_-(t_0) = \lim_{t \to t_0^-} f(t), \quad f_+(t_0) = \lim_{t \to t_0^+} f(t)$$

Examples:

1) The truncated Fourier series for the tent function, $tent_N(t)$ converge to $tent(t)$ for all t. In fact, it can be shown that the convergence is uniform, i.e. $\forall\, \varepsilon > 0 \,\exists\, N \, s.t. \, n \geq N \Rightarrow |tent(t) - tent_n(t)| < \varepsilon$ for all t at once.

2) The truncated Fourier series for the sawtooth function, $saw_N(t)$ converge to $saw(t)$ for all $t \neq \pi + 2k\pi, k \in \mathbb{Z}$ (i.e. everywhere except at the jump points). At these jump points the Fourier series converges to the average of the left and right hand limits of *saw*, which is 0. (In fact, each partial sum evaluates to 0 at those points.) The convergence at the other t values is pointwise, but not uniform, as the convergence takes longer nearer the jump points.)



$tent(t) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n\ odd} \frac{1}{n^2} \cos nt$

<u>Exercise 4)</u> We can derive "magic" summation formulas using Fourier series. (See your homework for some more.) From Theorem 2 we know that the Fourier series for $tent(t)$ converges for all t. In particular

$$0 = tent(0) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n\ odd} \frac{1}{n^2} \cos(n \cdot 0).$$

<u>4a</u>) Deduce

$$\frac{1}{1} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \sum_{n\ odd} \frac{1}{n^2} = \frac{\pi^2}{8}.$$

<u>4b</u>) Verify and use

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n\ odd} \frac{1}{n^2} + \sum_{n\ even} \frac{1}{n^2} = \sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{k^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n\ odd} \frac{1}{n^2} + \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} \Rightarrow \cancel{\tfrac{3}{4}} \sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n\ odd} \frac{1}{n^2} = \frac{\pi^2}{8} \frac{4}{3}$$

$$= \frac{\pi^2}{6}.$$

to show

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

4a) $\frac{\pi}{2} = \frac{4}{\pi} \sum_{n\ odd} \frac{1}{n^2} \Rightarrow \frac{\pi^2}{8} = \sum_{n = odd} \frac{1}{n^2}$

See final HW assignment.

this assignment will be used to replace a low score on HW. (your 3rd lowest)

Differentiating Fourier Series:

Theorem 3 Let f be 2π-periodic, piecewise differentiable and continuous, and with f' piecewise continuous. Let f have Fourier series

$$\frac{d}{dt}: \quad f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\,t) + \sum_{n=1}^{\infty} b_n \sin(n\,t).$$

Then f' has the Fourier series you'd expect by differentiating term by term:

differentiated term by term.

$$f' \sim \sum_{n=1}^{\infty} -n\,a_n \sin(n\,t) + \sum_{n=1}^{\infty} \boxed{n\,b_n} \cos(n\,t)$$

proof: Let f' have Fourier series

$$f' \sim \frac{A_0}{2} + \sum_{n=1}^{\infty} \boxed{A_n} \cos(n\,t) + \sum_{n=1}^{\infty} B_n \sin(n\,t).$$

Then

$$\boxed{A_n} = \frac{1}{\pi}\int_{-\pi}^{\pi} \overbrace{f'(t)\underbrace{\cos(n\,t)}_{u}\,\mathrm{d}t}^{dv},\ n \in \mathbb{N}.$$

Integrate by parts with $u = \cos(n\,t)$, $dv = f'(t)dt$, $du = -n\sin(n\,t)dt$, $v = f(t)$:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f'(t)\cos(n\,t)\ \mathrm{d}t = \frac{1}{\pi}\underbrace{f(t)}_{v}\underbrace{(\overset{\cos nt}{\cancel{-n)\sin(n\,t}})}_{u}\Big]_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(t)}_{v}\underbrace{(-n)\sin(n\,t)\ \mathrm{d}t}_{du}$$

$$= 0 + \frac{n}{\pi}\int_{-\pi}^{\pi} f(t)\sin(n\,t)\ \mathrm{d}t = \boxed{n\,b_n}$$

Similarly, $A_0 = 0$, $B_n = -n\,a_n$.

$\square$

Remark: This is analogous to what happened with Laplace transform. In that case, the transform of the derivative multiplied the transform of the original function by s (and there were correction terms for the initial values). In this case the tranformed variables are the a_n, b_n which depend on n. And the Fourier series "transform" of the derivative of a function multiplies these coefficients by n (and permutes them).

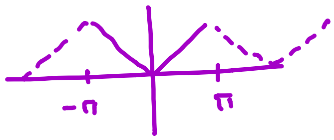
Note complex form of Fourier series.

$$f \sim \sum_n c_n e^{int} \qquad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt$$

$$f' \sim \sum_n c_n(in)e^{int}$$

you can work out that this is equivalent to the real form.

$|t|$:

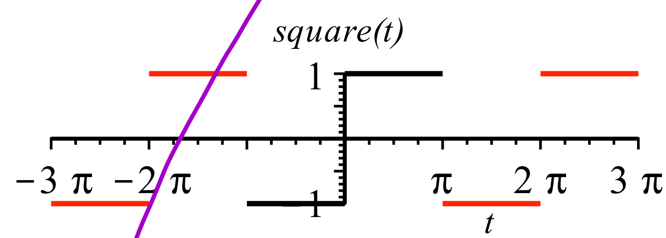


$$\frac{\pi}{2} - \frac{4}{\pi}\sum_{n \text{ odd}} \frac{1}{n^2}\cos nt$$
$$\|$$

Exercise 5a Use the differentiation theorem and the Fourier series for $tent(t)$ to find the Fourier series for the square wave, $square(t)$, which is the $2\pi-$ periodic extension of

$\frac{d}{dt}\left(tent(t)\right) = square(t)$

$$f(t) = \begin{cases} -1 & -\pi < t < 0 \\ 1 & 0 < t < \pi \end{cases}$$



(You will find the series directly from the definition in your homework.)
5b) Deduce the magic formula

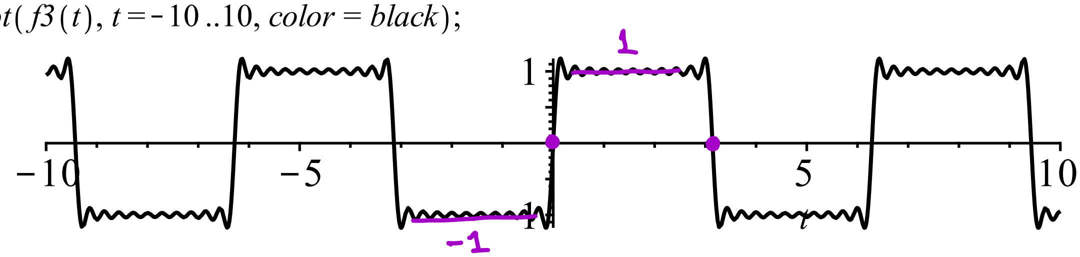
$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ... = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)} = \frac{\pi}{4}.$$

$$\frac{d}{dt}\Big(\qquad \Big) = 0 - \frac{4}{\pi}\sum_{n \text{ odd}} \frac{1}{n^2}\left(-\sin(nt)\, n\right)$$
$$= \frac{4}{\pi}\sum_{n \text{ odd}} \frac{1}{n}\sin nt$$

solution: $square \sim \frac{4}{\pi}\sum_{n\ odd} \frac{1}{n}\sin(n\,t)$

$> f3 := t \rightarrow \frac{4}{\pi} \cdot \sum_{n=0}^{10} \frac{1}{(2\cdot n+1)} \cdot \sin((2\cdot n+1)\cdot t):$

$> plot(f3(t), t = -10..10, color = black);$



$>$

Could you check the Fourier coefficients with technology?

Math 2280-001
Week 14, April 17-21 and Week 15, April 24:
9.1-9.4 Fourier series and forced oscillations revisited; 6.1-6.4 nonlinear autonomous systems.
Mon Apr 17

9.1-9.3 Fourier Series. (On Wednesday we'll revisit forced oscillations, section 9.4, and explain the results we obtained playing "the resonance game" with convolution integrals last Wednesday.)

- Finish Friday's notes if necessary. The key points to recall from Friday are that Fourier series are a way of expressing piecewise continuous functions *f* on the interval $[-\pi, \pi]$ (or equivalently, 2π-periodic functions *f*), as infinite sum of trigonometric functions. If *f* has Fourier series

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\,t) + \sum_{n=1}^{\infty} b_n \sin(n\,t)$$

average value of f

Then the partial sums

$$f_N(t) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(nt) + \sum_{n=1}^{N} b_n \sin(nt)$$

with

$$a_0 := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t \qquad \left(\frac{a_0}{2} = \left\langle f, \frac{1}{\sqrt{2}}\right\rangle \frac{1}{\sqrt{2}}\right)$$

$$\frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t$$

$$a_n := \langle f, \cos(n\,t)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(n\,t)\,\mathrm{d}t,\ n \in \mathbb{N}$$

$$b_n := \langle f, \sin(n\,t)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(n\,t)\,\mathrm{d}t,\ n \in \mathbb{N}$$

are the projections of *f* onto

$$V_N = span\left\{\frac{1}{\sqrt{2}}, \cos(t), \cos(2\,t), \ldots, \cos(N\,t), \sin(t), \sin(2\,t), \ldots \sin(N\,t)\right\}.$$

These partial sums converge to *f* as $N \to \infty$ in the ways described in Friday's notes.

- After finishing with Friday's notes, continue with today's...

Exercise 1 If a function ("vector") is already in a subspace, then projection onto that subspace leaves the function fixed. Use that fact to very quickly compute the 2π-periodic Fourier series for

a) $f(t) = \sin(5t) - 8\cos(10t)$ = Fourier series. $f \in V_{10} = \text{span}\{1, \cos t, \ldots \cos 10t, \sin t, \ldots \sin 10t\}$

b) $g(t) = \cos^2(3t)$

$$\frac{1 + \cos 6t}{2} = \frac{1}{2} + \frac{1}{2}\cos 6t$$

$$\cos^2\theta = \frac{\cos 2\theta + 1}{2} =$$

Fourier series for $2\,L$-periodic functions:

Theorem: Consider the vector space of piecewise continuous, $2\,L$-periodic functions. Then the inner product

$$\langle g, h\rangle := \frac{1}{L}\int_{-L}^{L} g(u)h(u)\,\mathrm{d}u$$

t: $[-\pi, \pi]$ $\quad u = \frac{L}{\pi}t$ $\quad t = \frac{\pi}{L}u$ $\quad u$: $[-L, L]$

makes

$$\left\{\frac{1}{\sqrt{2}}, \cos\left(\frac{\pi}{L}u\right), \cos\left(\frac{2\pi}{L}u\right), \ldots \cos\left(\frac{k\pi}{L}u\right), \ldots \sin\left(\frac{\pi}{L}u\right), \sin\left(\frac{2\pi}{L}u\right), \ldots\sin\left(\frac{k\pi}{L}u\right) \ldots\right\}$$

into an orthonormal collection of functions.

proof: The substitution $\frac{\pi}{L}u = t$, equivalently $u = \frac{L}{\pi}t$ converts between $2\,L$-periodic functions $g(u), h(u)$ and $2\,\pi$-periodic functions $g\left(\frac{L}{\pi}t\right), h\left(\frac{L}{\pi}t\right)$. Also (verify this!!!)

2L periodic

then
$\tilde{g}(t) = g\left(\frac{L}{\pi}t\right)$
$\tilde{h}(t) = h\left(\frac{L}{\pi}t\right)$
2π-periodic.

$$\frac{1}{L}\int_{-L}^{L} g(u)h(u)\,\mathrm{d}u = \frac{1}{\pi}\int_{-\pi}^{\pi} g\left(\frac{L}{\pi}t\right)h\left(\frac{L}{\pi}t\right)\,\mathrm{d}t\,.$$

$t = \frac{\pi}{L}u$ / $u = \frac{L}{\pi}t$
$dt = \frac{\pi}{L}du$ / $du = \frac{L}{\pi}dt$

□

Thus the Fourier series for a $2\,L$- periodic function f is defined by

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n\frac{\pi}{L}u\right) + \sum_{n=1}^{\infty} b_n \sin\left(n\frac{\pi}{L}u\right)$$

with

$$a_0 = \frac{1}{L}\int_{-L}^{L} g(u)\,\mathrm{d}u$$

$$a_n := \left\langle f, \cos\left(n\frac{\pi}{L}\,u\right)\right\rangle = \frac{1}{L}\int_{-L}^{L} f(u)\cos\left(n\frac{\pi}{L}\,u\right)\,\mathrm{d}u,\ \ n \in \mathbb{N}$$

$$b_n := \left\langle f, \sin\left(n\frac{\pi}{L}\,u\right)\right\rangle = \frac{1}{L}\int_{-L}^{L} f(u)\sin\left(\frac{\pi}{L}\,u\right)\,\mathrm{d}u,\ n \in \mathbb{N}$$

(and then we usually use the dummy variable t rather than u). As a result, Fourier series for $2\,L$-periodic functions along with convergence theorems, are "equivalent" to ones for $2\,\pi$-periodic ones, via this *isometry* of the two vector spaces.

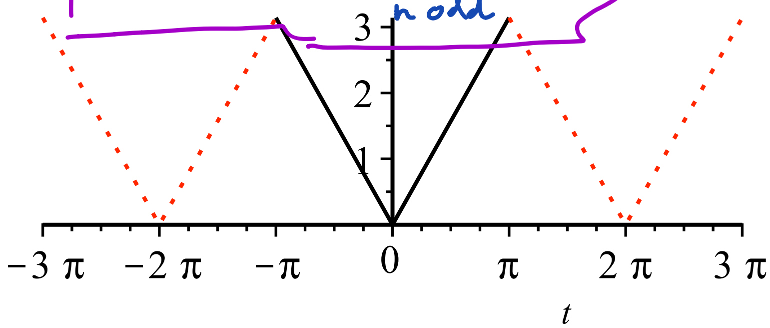
$f(u) = 2L$-periodic.
$f\left(\frac{L}{\pi}t\right) = 2\pi$-periodic. $\sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$

$u = \frac{L}{\pi}t$
$\frac{\pi}{L}u = t$

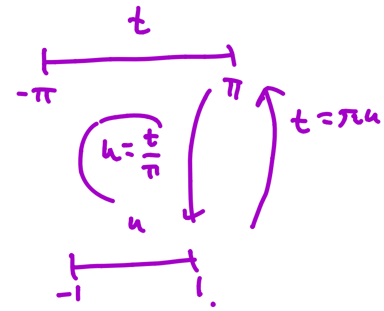
so $f(u) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\left(\frac{\pi}{L}u\right) + \sum_{n=1}^{\infty} b_n \sin n\left(\frac{\pi}{L}u\right)$ (a_n's, b_n's computed as before).

Exercise 2)

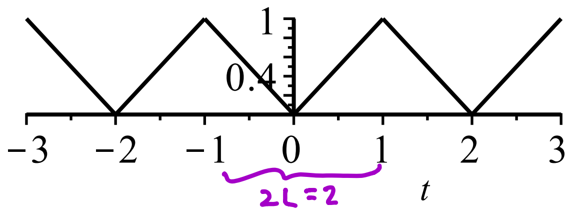
a) Use the Fourier series for $tent(t)$: $= \frac{\pi}{2} - \frac{4}{\pi}\sum_{n \text{ odd}} \frac{1}{n^2}\cos nt$



$$tent \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{n\ odd}\frac{1}{n^2}\cos(n\,t)$$



to deduce the Fourier series for the related function $f(u)$ with period 2 that has graph



$2L = 2$
$L = 1$

$$f(u) = f\left(\frac{t}{\pi}\right) = \frac{1}{\pi}\,tent(t) = \frac{1}{\pi}\left[\frac{\pi}{2} - \frac{4}{\pi}\sum_{n\text{ odd}}\frac{1}{n^2}\cos nt\right]$$

$$= \frac{1}{2} - \frac{4}{\pi^2}\sum_{n\text{ odd}}\frac{1}{n^2}\cos(n\pi u)$$

solution: $f \sim \frac{1}{2} - \frac{4}{\pi^2}\sum_{n\ odd}\frac{1}{n^2}\cos(n\,\pi\,u)$

We talked about differentiating Fourier series term by term in Friday's notes. There is also:

<u>Theorem</u> If f is piece-wise continuous, $2\,L$-periodic, with Fourier series

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n \frac{\pi}{L} t\right) + \sum_{n=1}^{\infty} b_n \sin\left(n \frac{\pi}{L} t\right)$$

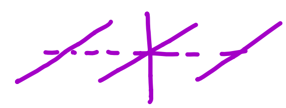
$$\int \cos \frac{n\pi}{L} t = \frac{L}{n\pi} \sin \frac{n\pi}{L} t$$

then the antiderivative may be computed by term by term antidifferentiation, and the corresponding series will converge for each t:

$$\int_0^t f(s)\, \mathrm{d}s = \frac{a_0}{2} t + \sum_{n=1}^{\infty} a_n \left(\frac{L}{n\,\pi}\right) \sin\left(n \frac{\pi}{L} t\right) + \sum_{n=1}^{\infty} b_n \left(\frac{L}{n\,\pi}\right)\left(-\cos\left(n \frac{\pi}{L} t\right) + 1\right).$$

<u>Exercise 3)</u> (This is the first part of your homework exercise 9.3.19)
Start with

"t"

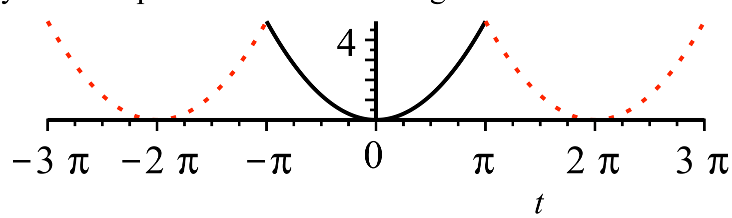


$$\int_0^t dt \qquad t = saw(t) \sim 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\,t)$$

and integrate to get the $2\,\pi$- periodic function that on $[-\pi, \pi]$ is given by

$$\boxed{\frac{t^2}{2} = \frac{\pi^2}{6} + 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos(n\,t).}$$

Hint: The value of the constant term is easiest to compute as $\frac{a_0}{2}$. If you compare to the definite integral formula in the Theorem you will reproduce one of the "magic" series.



In your homework you will antidifferentiate twice more to get a formula for the periodic extension of $g(t) = \frac{t^4}{24}$ and some more magic formulas.

$$\int_0^t saw(\tau)\, d\tau = 2 \int_0^t \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\tau \; d\tau$$

$$-\pi \le t \le \pi: \qquad \frac{t^2}{2} = 2 \sum_{n=1}^{\infty} \underbrace{\int_0^t \frac{(-1)^{n+1}}{n} \sin n\tau\, d\tau}_{\frac{(-1)^{n+1}}{n}\left[-\frac{\cos n\tau}{n}\right]_0^t}$$

$$= 2 \sum_{n=1}^{\infty} (-1)^n \left(\frac{\cos nt}{n^2} - \frac{1}{n^2}\right)$$

$$\boxed{\frac{t^2}{2} = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} + 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nt}$$