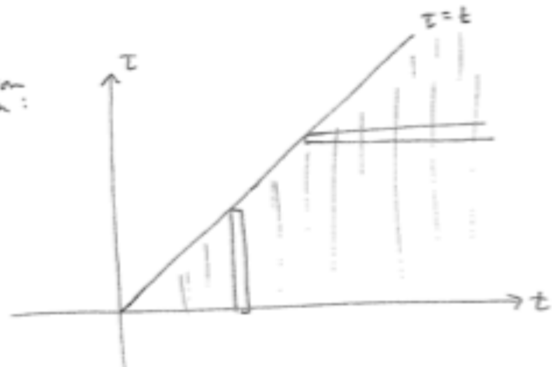
proof of convolution theorem:

(is a good review of iterated integrals)

$$\mathcal{L}\{f*g\}(s) = \int_0^\infty e^{-st}\left(\int_0^t f(\tau)\,g(t-\tau)\,d\tau\right)dt$$

$$= \int_0^\infty \int_0^t e^{-st} f(\tau)\,g(t-\tau)\,d\tau\,dt$$

integration region:



interchange limits:

$$= \int_0^\infty \int_\tau^\infty e^{-st} f(\tau)\,g(t-\tau)\,dt\,d\tau$$

$$= \int_0^\infty \int_\tau^\infty e^{-s\tau} f(\tau)\; e^{-s(t-\tau)} g(t-\tau)\,dt\,d\tau \qquad \text{(pattern recognition)}$$

$$= \int_0^\infty e^{-s\tau} f(\tau)\left[\int_\tau^\infty e^{-s(t-\tau)} g(t-\tau)\,dt\right] d\tau$$

$$\tilde{t} = t-\tau$$

$$d\tilde{t} = dt$$

$$\underbrace{\left[\int_0^\infty e^{-s\tilde{t}}\, g(\tilde{t})\,d\tilde{t}\right]}_{G(s)}$$

$$= G(s)\int_0^\infty e^{-s\tau} f(\tau)\,d\tau$$

$$= G(s)\,F(s) \qquad !!$$

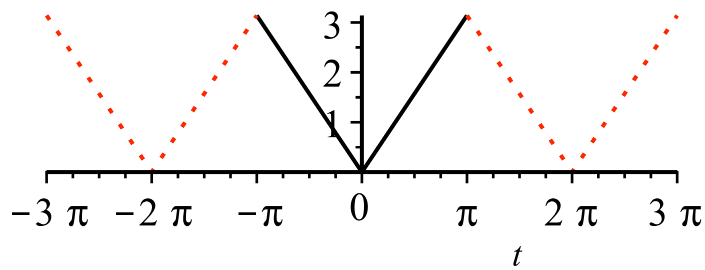
• Rest of class notes are posted.

Math 2280-001
Fri Apr 14

Chapter 9 Fourier Series and applications to differential equations (and partial differential equations)
9.1-9.2 Fourier series definition and convergence.

The idea of Fourier series is related to the linear algebra concepts of dot product, norm, and projection.
We'll review this connection after the definition of Fourier series:

Let $f: [-\pi, \pi] \to \mathbb{R}$ be a piecewise continuous function, or equivalently, extend to $f: \mathbb{R} \to \mathbb{R}$ as a 2π-periodic function.
Example one could consider the 2π-periodic extension of $f(t) = |t|$, initially defined on the t-interval $[-\pi, \pi]$, to all of $\mathbb{R}$. Its graph is the so-called "tent function", $tent(t)$



The Fourier coefficents of a 2π- periodic function f are computed via the definitions

$$a_0 := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\, dt$$

$$a_n := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(n\,t)\, dt,\ n \in \mathbb{N}$$

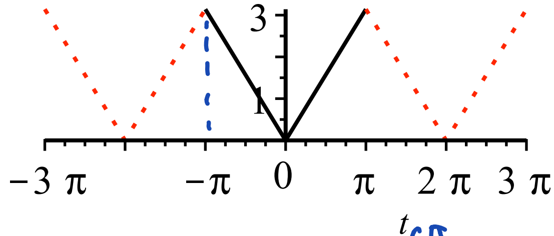
$$b_n := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(n\,t)\, dt,\ n \in \mathbb{N}$$

And the Fourier series for f is given by

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\,t) + \sum_{n=1}^{\infty} b_n \sin(n\,t).$$

The idea is that the partial sums of the Fourier series of f should actually converge to f. The reasons why this should be true combine linear algebra ideas related to orthonormal basis vectors and projection, with analysis ideas related to convergence. Let's do an example to illustrate the magic, before discussing (parts of) why the convergence actually happens.

Exercise 1 Consider the even function $f(t) = |t|$ on the interval $-\pi \le t \le \pi$, extended to be the 2π-periodic "tent function" $tent(t)$ of page 1. Find the Fourier coefficients a_0, a_n, b_n and the Fourier series for *tent*. The answer is below, along with a graph of partial sum of the Fourier series.



$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{|t|\sin(nt)}_{\text{odd fcn}}\,dt = 0\,!$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |t|\,dt = \frac{2}{\pi}\underbrace{\int_0^{\pi} t\,dt}_{\underbrace{\frac{1}{2}t^2\Big]_0^{\pi}}_{\frac{\pi^2}{2}}}$$

$$a_0 = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |t|\cos nt\,dt = \frac{2}{\pi}\int_0^{\pi} \underset{=u}{t}\,\underset{=dv}{\cos nt\,dt} \qquad v = +\frac{\sin(nt)}{n}$$

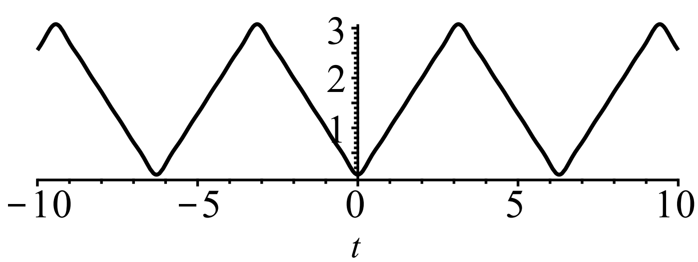
$$a_n = \frac{2}{\pi}\left[\cancel{t\left(\frac{\sin nt}{n}\right)\Big|_{t=0}^{\pi}} \to 0 - \int_0^{\pi}\frac{\sin(nt)}{n}\,dt\right]$$

$$a_n = \frac{2}{\pi}\left[-\,-\frac{\cos nt}{n^2}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{2}{\pi n^2}\left[\cos(n\pi) - 1\right] = \begin{cases} 0 & n \text{ even} \\ -\frac{4}{\pi n^2}, & n \text{ odd} \end{cases}$$

solution: $tent \sim \underset{\uparrow\ \frac{a_0}{2}}{\frac{\pi}{2}} - \frac{4}{\pi}\sum_{n\ odd}\frac{1}{n^2}\cos(n\,t)$

> $f1 := t \to \frac{\pi}{2} - \frac{4}{\pi}\cdot\sum_{j=0}^{4}\frac{1}{(2\cdot j+1)^2}\cos((2j+1)\cdot t)$:

> $plot(f1(t), t=-10..10, color=black)$;



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Using technology to compute Fourier coefficients:

> $f := t \rightarrow |t|;$

$$f := t \rightarrow |t| \tag{2}$$

> $a0 := \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} f(t)\, dt;$

$assume(n, integer);$ *# this will let Maple attempt to evaluate the integrals*

$a := n \rightarrow \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} f(t) \cdot \cos(n \cdot t)\, dt:$

$b := n \rightarrow \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} f(t) \cdot \sin(n \cdot t)\, dt:$

$a(n);$
$b(n);$

I believe Wolfram alpha can do this too

$$a0 := \pi$$

$$\frac{2\left((-1)^{n\sim} - 1\right)}{\pi\, n\sim^2}$$

$$0 \tag{3}$$

>

So what's going on?
Recall the ideas of dot product, angle, orthonormal basis and projection onto subspaces, in $\mathbb{R}^N$, from linear algebra:

For $\underline{\boldsymbol{x}}, \underline{\boldsymbol{y}} \in \mathbb{R}^N$, the dot product $\underline{\boldsymbol{x}} \cdot \underline{\boldsymbol{y}} := \sum_{k=1}^{N} x_k\, y_k$ satisfies for all vectors $\underline{\boldsymbol{x}}, \underline{\boldsymbol{y}}, \underline{\boldsymbol{z}} \in \mathbb{R}^N$ and scalars $s \in \mathbb{R}$:

a) $\underline{\boldsymbol{x}} \cdot \underline{\boldsymbol{x}} \geq 0$ and $= 0$ if and only if $\underline{\boldsymbol{x}} = \underline{\boldsymbol{0}}$
b) $\underline{\boldsymbol{x}} \cdot \underline{\boldsymbol{y}} = \underline{\boldsymbol{y}} \cdot \underline{\boldsymbol{x}}$
c) $\underline{\boldsymbol{x}} \cdot (\underline{\boldsymbol{y}} + \underline{\boldsymbol{z}}) = \underline{\boldsymbol{x}} \cdot \underline{\boldsymbol{y}} + \underline{\boldsymbol{x}} \cdot \underline{\boldsymbol{z}}$
d) $(s\,\underline{\boldsymbol{x}}) \cdot \underline{\boldsymbol{y}} = s(\underline{\boldsymbol{x}} \cdot \underline{\boldsymbol{y}}) = \underline{\boldsymbol{x}} \cdot (s\,\underline{\boldsymbol{y}})$

From these four properties one can define the *norm* or magnitude of a vector by

$$\|\underline{\boldsymbol{x}}\| = \sqrt{\underline{\boldsymbol{x}} \cdot \underline{\boldsymbol{x}}}$$

and the distance bewteen two vectors $\underline{\boldsymbol{x}}, \underline{\boldsymbol{y}}$ by

$$dist(\underline{\boldsymbol{x}}, \underline{\boldsymbol{y}}) := \|\underline{\boldsymbol{x}} - \underline{\boldsymbol{y}}\|.$$

One can check with algebra that the *Cauchy-Schwarz inequality* holds:

$$|\underline{\boldsymbol{x}} \cdot \underline{\boldsymbol{y}}| \leq \|\underline{\boldsymbol{x}}\|\ \|\underline{\boldsymbol{y}}\|\ ,$$

with equality if and only if $\underline{\boldsymbol{x}}, \underline{\boldsymbol{y}}$ are scalar multiples of each other. One consequence of the Cauchy-Schwarz inequality is the *triangle inequality*

$$\|\underline{\boldsymbol{x}} + \underline{\boldsymbol{y}}\| \leq \|\underline{\boldsymbol{x}}\| + \|\underline{\boldsymbol{y}}\|,$$

with equality if and only if $\underline{\boldsymbol{x}}, \underline{\boldsymbol{y}}$ are non-negative scalar multiples of each other. Equivalently, in terms of Euclidean distance,

$$dist(\underline{\boldsymbol{x}}, \underline{\boldsymbol{z}}) \leq dist(\underline{\boldsymbol{x}}, \underline{\boldsymbol{y}}) + dist(\underline{\boldsymbol{y}}, \underline{\boldsymbol{z}})\ .$$

Another consequence of the Cauchy-Schwarz inequality is that one can define the angle θ between $\underline{\boldsymbol{x}}, \underline{\boldsymbol{y}}$ via

$$\cos(\theta) := \frac{\underline{\boldsymbol{x}} \cdot \underline{\boldsymbol{y}}}{\|\underline{\boldsymbol{x}}\|\ \|\underline{\boldsymbol{y}}\|}\ ,$$

for $0 \leq \theta \leq \pi$, because $-1 \leq \dfrac{\underline{\boldsymbol{x}} \cdot \underline{\boldsymbol{y}}}{\|\underline{\boldsymbol{x}}\|\ \|\underline{\boldsymbol{y}}\|} \leq 1$ holds so that θ exists. In particular two vectors $\underline{\boldsymbol{x}}, \underline{\boldsymbol{y}}$ are perpendicular, or *orthogonal* if and only if

$$\underline{\boldsymbol{x}} \cdot \underline{\boldsymbol{y}} = 0.$$

If one has a $n-$ *dimensional* subspace $W \subseteq \mathbb{R}^N$ an *orthonormal basis* $\{\underline{\boldsymbol{u}}_1, \underline{\boldsymbol{u}}_2, \dots \underline{\boldsymbol{u}}_n\}$ for W is a collection of unit vectors (normalized to length 1), which are also mutually orthogonal. (One can find such bases via the Gram-Schmidt algorithm.) For such an ortho-normal basis the nearest point projection of a vector $\underline{\boldsymbol{x}} \in \mathbb{R}^N$ onto W is given by

$$proj_W \underline{\boldsymbol{x}} = (\underline{\boldsymbol{x}} \cdot \underline{\boldsymbol{u}}_1)\underline{\boldsymbol{u}}_1 + (\underline{\boldsymbol{x}} \cdot \underline{\boldsymbol{u}}_2)\underline{\boldsymbol{u}}_2 + \dots + (\underline{\boldsymbol{x}} \cdot \underline{\boldsymbol{u}}_n)\underline{\boldsymbol{u}}_n = \sum_{k=1}^{n} (\underline{\boldsymbol{x}} \cdot \underline{\boldsymbol{u}}_k)\underline{\boldsymbol{u}}_k\ .$$

For any $\underline{\boldsymbol{x}}$ (already) in W, $proj_W \underline{\boldsymbol{x}} = \underline{\boldsymbol{x}}$.

The entire algebraic/geometric development on the previous page just depended on the four algebraic properties **a,b,c,d** for the dot product. So it can be generalized:

Definition Let V is any real-scalar vector space. we call V an inner product space if there is an inner product $\langle f, g\rangle$
for which the inner product satisfies $\forall\, f, g, h \in V$ and scalars $s \in \mathbb{R}$:

a) $\langle f, f\rangle \geq 0$. $\langle f, f\rangle = 0$.

b) $\langle f, g\rangle = \langle g, f\rangle$.

c) $\langle f, g+h\rangle = \langle f, g\rangle + \langle f, h\rangle$

d) $\langle (s\,f), g\rangle = s\langle f, g\rangle = \langle f, s\,g\rangle$.

$$\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)g(t)\,dt$$

In this case one can define $\|f\| = \sqrt{\langle f, f\rangle}$, $dist(f, g) = \|f - g\|$; prove the Cauchy-Schwarz inequality and the triangle inequalities; define angles between vectors, and in particular, orthogonality between vectors; find ortho-normal bases $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots \boldsymbol{u}_n\}$ for finite-dimensional subspaces W, and prove that for any $f \in V$ the nearest element in W to f is given by

$$proj_W f = \langle f, u_1\rangle u_1 + \langle f, u_2\rangle u_2 + \ldots + \langle f, u_n\rangle u_n = \sum_{k=1}^{n} < f, u_k > u_k .$$

Theorem Let $V = \{f : \mathbb{R} \to \mathbb{R}\ s.t.\ f$ is piecewise continuous and 2π-periodic$\}$. Define

$$\langle f, g\rangle := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)g(t)\,\mathrm{d}t.$$

1) Then V, $\langle\ ,\ \rangle$ is an inner product space.

2) Let $V_N := span\left\{\frac{1}{\sqrt{2}}, \cos(t), \cos(2\,t), \ldots, \cos(N\,t), \sin(t), \sin(2\,t), \ldots \sin(N\,t)\right\}$. Then the $2\,N+1$ functions listed in this collection are an orthonormal basis for the $(2\,N+1)$ dimensional subspace V_N. In particular, for any $f \in V$ the nearest function in V_N to f is given by

$$proj_{V_N} f = \underbrace{< f, \frac{1}{\sqrt{2}} > \frac{1}{\sqrt{2}}}_{\langle f, 1\rangle \frac{1}{2}} + \sum_{n=1}^{N} < f, \cos(nt) > \cos(nt) + \sum_{n=1}^{N} < f, \sin(nt) > \sin(nt)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(nt) + \sum_{n=1}^{N} b_n \sin(nt)$$

where a_0, a_n, b_n are the Fourier coefficients defined on page 1.

<u>Exercise 2)</u> Check that $\left\{\frac{1}{\sqrt{2}}, \cos(t), \cos(2t), ..., \cos(N t), \sin(t), \sin(2t), ... \sin(N t)\right\}$ are orthonormal with respect to the inner product

$$\langle f, g\rangle := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)g(t)\,\mathrm{d}t$$

Hint:

$$\cos((m+k)t) = \cos(m t)\cos(k t) + \sin(m t)\sin(k t)$$
$$\sin((m+k)t) = \sin(m t)\cos(k t) + \cos(m t)\sin(k t)$$

so

$$\cos(m t)\cos(k t) = \frac{1}{2}(\cos((m+k)t) + \cos((m-k)t)) \quad \text{(even if } m = k) \quad *$$

$$\sin(m t)\sin(k t) = \frac{1}{2}(\cos((m-k)t) - \cos((m+k)t)) \quad \text{(even if } m = k)$$

$$\cos(m t)\sin(k t) = \frac{1}{2}(\sin((m+k)t) + \sin((-m+k)t))$$

e.g. $\langle \cos mt, \cos kt\rangle = 0 \quad k \neq m.$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}(\cos mt)(\cos kt)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\cos((m+k)t) + \cos((m-k)t)}{2}\,dt$$

$$= \frac{1}{\pi}\left[\frac{\sin((m+k)t)}{2(m+k)} + \frac{\sin((m-k)t)}{2(m-k)}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\cdot 0 = 0$$

$m=k$: is $\|\cos mt\| = 1$?

$$\|\cdot\|^2 = \frac{1}{\pi}\int_{-\pi}^{\pi}(\cos mt)^2\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1+\cos(2mt)}{2}\,dt$$

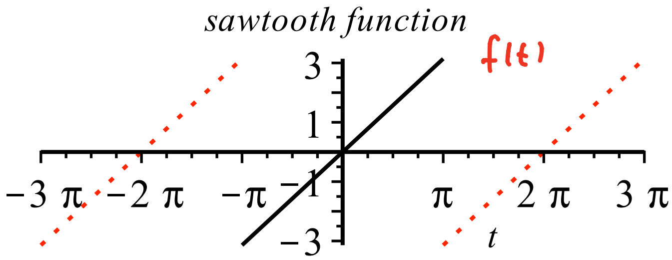
$$= \frac{1}{\pi}\left[\frac{t}{2} + \frac{\sin(2mt)}{4m}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\pi + 0\right] = 1 \checkmark$$

Exercise 3) Consider the $2\pi-$periodic odd function $saw(t)$ define by extending

$$f(t) = t, \quad -\pi < t \le \pi$$

as a $2\pi-$periodic function.



Find the Fourier series for $saw(t)$. Hint: you noticed that for the *even* tent function in Exercise 1 the *sine* Fourier coefficients were all zero. Which ones will be zero for any *odd* function? Why?

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{t\cos nt}_{\text{odd fcn}}\,dt = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{t\sin nt}_{\text{even fcn}}\,dt = \frac{2}{\pi}\int_0^{\pi} \underset{u}{t}\ \underbrace{\sin(nt)\,dt}_{dv} =$$

$$du = dt \quad v = -\frac{\cos nt}{n}$$

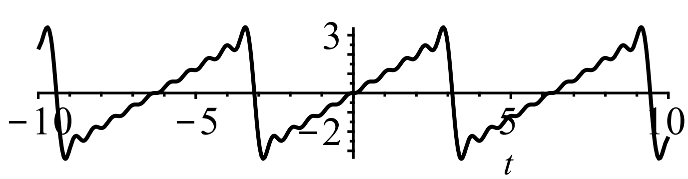
$$= \frac{2}{\pi}\left[t\left(-\frac{\cos nt}{n}\right)\Big]_0^{\pi} - \cancel{\int_0^{\pi} -\frac{\cos nt}{n}\,dt}\right] \to 0.$$

$$= \frac{2}{\cancel{\pi}}\left[\frac{\cancel{\pi}}{n}\left(-(-1)^n\right)\right] = \frac{2}{n}(-1)^{n+1}$$

solution: $saw \sim 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin(n\,t)$

> $f2 := t \to 2 \cdot \sum_{n=1}^{10} \frac{(-1)^{n+1}}{n} \cdot \sin(n \cdot t)$:

> $plot(f2(t), t = -10..10, color = black)$;



Convergence Theorems (These require some careful mathematical analysis to prove - they are often discussed in Math 5210, for example.)

Theorem 1 Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be 2π-periodic and piecewise continuous. Let

$$f_N = proj_{V_N} f = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(nt) + \sum_{n=1}^{N} b_n \sin(nt)$$

be the Fourier series truncated at N. Then

$$\lim_{n \rightarrow \infty} \|f - f_N\| = \lim_{n \rightarrow \infty} \left[\frac{1}{\pi} \int_{-\pi}^{\pi} \left(f(t) - f_N(t)\right)^2 \mathrm{d}t \right]^{\frac{1}{2}} = 0.$$

In other words, the distance between f_N and f converges to zero, where we are using the distance function that we get from the inner product,

$$dist(f, g) = \|f - g\| = \sqrt{\langle f - g, f - g \rangle} = \left[\frac{1}{\pi} \int_{-\pi}^{\pi} \left(f(t) - g(t)\right)^2 \mathrm{d}t \right]^{\frac{1}{2}}.$$

Theorem 2 If f is as in Theorem 1, and is (also) piecewise differentiable with at most jump discontinuities, then

(i) for any t_0 such that f is differentiable at t_0

$$\lim_{N \rightarrow \infty} f_N(t_0) = f(t_0) \quad \text{(pointwise convergence).}$$

(ii) for any t_0 where f is not differentiable (but is either continuous or has a jump discontinuity), then

$$\lim_{N \rightarrow \infty} f_N(t_0) = \frac{1}{2}\left(f_-(t_0) + f_+(t_0)\right)$$

where

$$f_-(t_0) = \lim_{t \rightarrow t_0^-} f(t), \quad f_+(t_0) = \lim_{t \rightarrow t_0^+} f(t)$$

Examples:

1) The truncated Fourier series for the tent function, $tent_N(t)$ converge to $tent(t)$ for all t. In fact, it can be shown that the convergence is uniform, i.e. $\forall\, \varepsilon > 0 \,\exists\, N\, s.t.\, n \geq N \Rightarrow |tent(t) - tent_n(t)| < \varepsilon$ for all t at once.

2) The truncated Fourier series for the sawtooth function, $saw_N(t)$ converge to $saw(t)$ for all $t \neq \pi + 2k\pi, k \in \mathbb{Z}$ (i.e. everywhere except at the jump points). At these jump points the Fourier series converges to the average of the left and right hand limits of saw, which is 0. (In fact, each partial sum evaluates to 0 at those points.) The convergence at the other t values is pointwise, but not uniform, as the convergence takes longer nearer the jump points.)

Exercise 4) We can derive "magic" summation formulas using Fourier series. (See your homework for some more.) From Theorem 2 we know that the Fourier series for $tent(t)$ converges for all t. In particular

$$0 = tent(0) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{n^2} \cos(n \cdot 0).$$

4a) Deduce

$$\frac{1}{1} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \sum_{n \text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{8}.$$

4b) Verify and use

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n \text{ odd}} \frac{1}{n^2} + \sum_{n \text{ even}} \frac{1}{n^2}$$

$$= \sum_{n \text{ odd}} \frac{1}{n^2} + \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

to show

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Differentiating Fourier Series:

Theorem 3 Let f be 2π-periodic, piecewise differentiable and continuous, and with f' piecewise continuous. Let f have Fourier series

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\,t) + \sum_{n=1}^{\infty} b_n \sin(n\,t).$$

Then f' has the Fourier series you'd expect by differentiating term by term:

$$f' \sim \sum_{n=1}^{\infty} -n\,a_n \sin(n\,t) + \sum_{n=1}^{\infty} n\,b_n \cos(n\,t)$$

proof: Let f' have Fourier series

$$f' \sim \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n\,t) + \sum_{n=1}^{\infty} B_n \sin(n\,t).$$

Then

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(t)\cos(n\,t)\,\mathrm{d}t,\ n \in \mathbb{N}.$$

Integrate by parts with $u = \cos(n\,t)$, $dv = f'(t)dt$, $du = -n\sin(n\,t)dt$, $v = f(t)$:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f'(t)\cos(n\,t)\,\mathrm{d}t = \frac{1}{\pi} f(t)(-n)\sin(nt)\,\Big]_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)(-n)\sin(n\,t)\,\mathrm{d}t$$

$$= 0 + \frac{n}{\pi}\int_{-\pi}^{\pi} f(t)\sin(n\,t)\,\mathrm{d}t = n\,b_n.$$

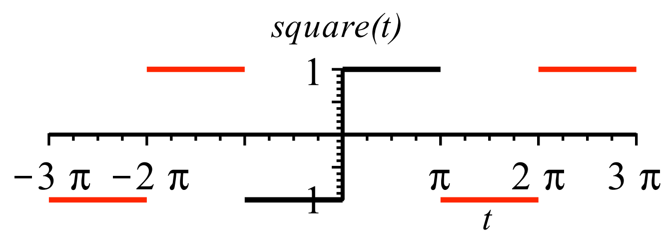
Similarly, $A_0 = 0$, $B_n = -n\,a_n$.

$\square$

Remark: This is analogous to what happened with Laplace transform. In that case, the transform of the derivative multiplied the transform of the original function by s (and there were correction terms for the initial values). In this case the tranformed variables are the a_n, b_n which depend on n. And the Fourier series "transform" of the derivative of a function multiplies these coefficients by n (and permutes them).

Exercise 5a Use the differentiation theorem and the Fourier series for $tent(t)$ to find the Fourier series for the square wave, $square(t)$, which is the $2\pi-$ periodic extension of

$$f(t) = \begin{cases} -1 & -\pi < t < 0 \\ 1 & 0 < t < \pi \end{cases}$$



(You will find the series directly from the definition in your homework.)

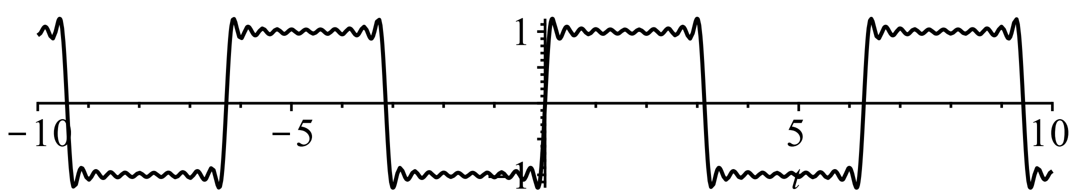
5b) Deduce the magic formula

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ... = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)} = \frac{\pi}{4}.$$

solution: $square \sim \frac{4}{\pi} \sum_{n\ odd} \frac{1}{n} \sin(n\,t)$

> $f3 := t \to \frac{4}{\pi} \cdot \sum_{n=0}^{10} \frac{1}{(2 \cdot n + 1)} \cdot \sin((2 \cdot n + 1) \cdot t) :$

> $plot(f3(t), t = -10..10, color = black);$



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Could you check the Fourier coefficients with technology?