Setup: an ~~under employed~~ dedicated mathematician/engineer/scientist
(your choice)

likes to take his/her child to the swings...

$L\theta'' + g\sin\theta = 0$

recall pendulum (linearized) eqtn, without forcing, for $\theta = \theta(t)$

$L\theta'' + g\theta = 0$

$\dashrightarrow \; x'' + g\frac{x}{L} = 0$

$\dashrightarrow \; m x'' + \frac{mg}{L} x = F_0 \cos\omega t$ ← parent forcing (?!)

$x(t) = L\sin\theta(t)$
$\approx L\theta$ for small θ

so $x'' \approx L\theta''$

$\dashrightarrow \; x'' + \frac{g}{L} x = \frac{F_0}{m}\cos\omega t$

$x'' + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$

parent pushes sinusoidally for exactly 5 cycles, and with $\frac{F_0}{m} = .2$ and then releases:

for resonance $\underline{\omega = \omega_0}$
construct swing with $L = "g" \approx 9.8$ m.
so $\omega_0^2 = 1$, $\underline{T_0 = 2\pi \approx 6.2}$ seconds

$x'' + x = \frac{F_0}{m}\cos t$ ☺

<u>Exercise 3a)</u> Explain why the description above leads to the differential equation initial value problem for $x(t)$

$$x''(t) + x(t) = .2\cos(t)\left(1 - u(t - 10\pi)\right) = .2\cos t - .2\cancel{\cos t}\,u(t-10\pi)$$
$$\cos(t-10\pi)$$
$$x(0) = 0$$
$$x'(0) = 0$$

$f(t) = \cos t$

$F(s) = \frac{s}{s^2+1}$

<u>3b)</u> Find $x(t)$. Show that after the parent stops pushing, the child is oscillating with an amplitude of exactly π meters (in our linearized model).

$$\frac{F_0}{m} = \begin{cases} .2\cos t & 0 \le t \le 10\pi \\ 0 & t > 10\pi \end{cases}$$

$$= .2\cos t\,(1 - u(t - 10\pi))$$

$$\mathcal{L}: \quad s^2 X(s) + X(s) = .2\frac{s}{s^2+1} - .2e^{-10\pi s}\frac{s}{s^2+1}$$

$$X(s) = .2\frac{s}{(s^2+1)^2} - .2e^{-10\pi s}\underbrace{\frac{s}{(s^2+1)^2}}$$

$$x(t) = .2\tfrac{1}{2}t\sin t$$

$\downarrow \mathcal{L}^{-1}: \frac{1}{2}t\sin t = f(t)$

$a = 10\pi$

$$-.2\tfrac{1}{2}u(t-10\pi)(t-10\pi)\sin(t-10\pi)$$

$\frac{1}{2k}t\sin kt$	$\frac{s}{(s^2+k^2)^2}$

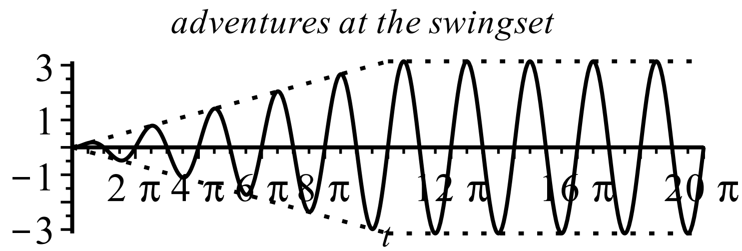
this factor was missing on the blackboard.

$$x(t) = \begin{cases} .1t\sin t & 0 \le t < 10\pi \\ \cancel{.1t\sin t} - .1(\cancel{t} - 10\pi)\sin(t - 10\pi) & t \ge 10\pi \\ = \pi\sin t & \end{cases}$$

$\sin(t-10\pi) = \sin t$ ✓

Pictures for the swing:

```
> plot1 := plot(.1·t·sin(t), t = 0..10·Pi, color = black) :
  plot2 := plot(Pi·sin(t), t = 10·Pi..20·Pi, color = black) :
  plot3 := plot(Pi, t = 10·Pi..20·Pi, color = black, linestyle = 2) :
  plot4 := plot(−Pi, t = 10·Pi..20·Pi, color = black, linestyle = 2) :
  plot5 := plot(.1·t, t = 0..10·Pi, color = black, linestyle = 2) :
  plot6 := plot(−.1·t, t = 0..10·Pi, color = black, linestyle = 2) :
  display({plot1, plot2, plot3, plot4, plot5, plot6}, title = `adventures at the swingset`);
```



Alternate approach via Chapter 3:

step 1) solve

$$x''(t) + x(t) = .2\cos(t)$$
$$x(0) = 0$$
$$x'(0) = 0$$

for $0 \le t \le 10\pi$.

step 2) Then solve

$$y''(t) + y(t) = 0$$
$$y(0) = x(10\pi)$$
$$y'(0) = x'(10\pi)$$

and set $x(t) = y(t-10)$ for $t > 10$.

EP 7.6 impulse functions and the δ operator.

Consider a force $f(t)$ acting on an object for only on a very short time interval $a \le t \le a + \varepsilon$, for example as when a bat hits a ball. This impulse p of the force is defined to be the integral

$$p := \int_a^{a+\varepsilon} f(t)\, dt$$

and it measures the net change in momentum of the object since by Newton's second law

$$m\, v'(t) = f(t)$$

$$\Rightarrow \int_a^{a+\varepsilon} m\, v'(t)\, dt = \int_a^{a+\varepsilon} f(t)\, dt = p$$

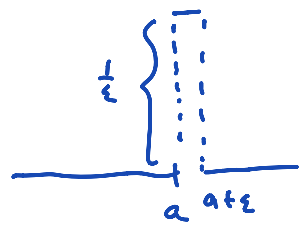
$$\Rightarrow m\, v(t)\Big]_{t=a}^{a+\varepsilon} = p\,.$$

Since the impulse p only depends on the integral of $f(t)$, and since the exact form of f is unlikely to be known in any case, the easiest model is to replace f with a constant force having the same total impulse, i.e. to set

$$f = p\, d_{a,\varepsilon}(t)$$

where $d_{a,\varepsilon}(t)$ is the unit impulse function given by

$$d_{a,\varepsilon}(t) = \begin{cases} 0, & t < a \\ \dfrac{1}{\varepsilon}, & a \le t < a + \varepsilon \\ 0, & t \ge a + \varepsilon \end{cases}.$$

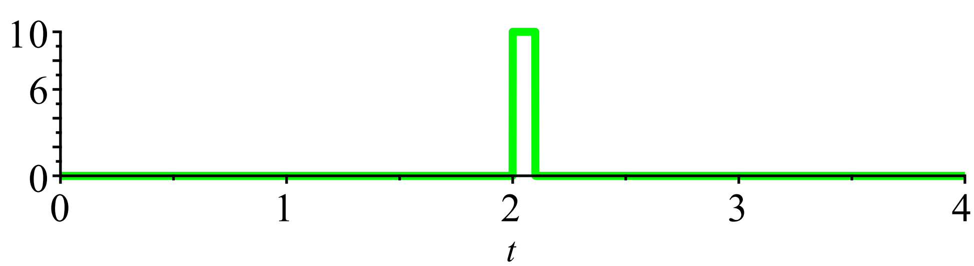


$\frac{1}{\varepsilon} u(t-a) - \frac{1}{\varepsilon} u(t-(a+\varepsilon))$

Notice that

$$\int_a^{a+\varepsilon} d_{a,\varepsilon}(t)\, dt = \int_a^{a+\varepsilon} \frac{1}{\varepsilon}\, dt = 1\,.$$

Here's a graph of $d_{2,.1}(t)$, for example:



Since the unit impulse function is a linear combination of unit step functions, we could solve differential equations with impulse functions so-constructed. As far as Laplace transform goes, it's even easier to take the limit as $\varepsilon \to 0$ for the Laplace transforms $\mathcal{L}\{d_{a,\varepsilon}(t)\}(s)$, and this effectively models impulses on very short time scales.

$$d_{a,\varepsilon}(t) = \frac{1}{\varepsilon}\left[u(t-a) - u(t-(a+\varepsilon))\right]$$

$$\frac{d}{d\varepsilon}\, 1-e^{-\varepsilon s} = -e^{-\varepsilon s}(-s) = s e^{-\varepsilon s}$$

$$\Rightarrow \mathcal{L}\left\{d_{a,\,\varepsilon}(t)\right\}(s) = \frac{1}{\varepsilon}\left(\frac{e^{-a\,s}}{s} - \frac{e^{-(a\,+\,\varepsilon)s}}{s}\right)$$

$$= e^{-a\,s}\left(\frac{1-e^{-\varepsilon\,s}}{\varepsilon\,s}\right).$$

@ $\varepsilon=0$ $\left(\frac{1-e^{-\varepsilon s}}{\varepsilon s}\right) = \frac{0}{0}$

In Laplace land we can use L'Hopital's rule (in the variable ε) to take the limit as $\varepsilon \rightarrow 0$:

$$\lim_{\varepsilon\,\rightarrow\,0} e^{-a\,s}\left(\frac{1-e^{-\varepsilon\,s}}{\varepsilon\,s}\right) = e^{-a\,s}\lim_{\varepsilon\,\rightarrow\,0}\left(\frac{\cancel{s}\,e^{-\varepsilon\,s}}{\cancel{s}}\right) = e^{-a\,s}.$$

L'Hopital

The result in time t space is not really a function but we call it the "delta function" $\delta(t-a)$ anyways, and visualize it as a function that is zero everywhere except at $t = a$, and that it is infinite at $t = a$ in such a way that its integral over any open interval containing a equals one. As explained in EP7.6, the delta "function" can be thought of in a rigorous way as a linear transformation, not as a function. It can also be thought of as the derivative of the unit step function $u(t-a)$, and this is consistent with the Laplace table entries for derivatives of functions. In any case, this leads to the very useful Laplace transform table entry

$\delta(t-a)$ unit impulse function (delta function)	$e^{-a\,s}$	for impulse forcing

Exercise 4) Revisit the swing. In this case the parent is providing an impulse each time the child passes through equilibrium position after completing a cycle.

$$x''(t) + x(t) = .2\,\pi\left[\delta(t) + \delta(t-2\pi) + \delta(t-4\pi) + \delta(t-6\pi) + \delta(t-8\pi)\right]$$
$$x(0) = 0$$
$$x'(0) = 0\,.$$

$x(0)$
$x'(0) = 0$

$$\mathcal{L}:\quad X(s)(s^2+1) = .2\pi\left[1 + e^{-2\pi s} + e^{-4\pi s} + e^{-6\pi s} + e^{-8\pi s}\right]$$

$$X(s) = .2\pi\left[\frac{1}{s^2+1} + e^{-2\pi s}\cdot\frac{1}{s^2+1} + \dots + e^{-8\pi s}\cdot\frac{1}{s^2+1}\right]$$

$f(t)$

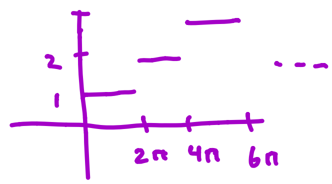
$$x(t) = .2\pi\left\{\sin t + u(t-2\pi)\underbrace{\sin(t-2\pi)}_{} + u(t-4\pi)\underbrace{\sin(t-4\pi)}_{\sin t} + \dots\right\}$$

$$x(t) = .2\pi\left[\sin t\right]\left[1 + u(t-2\pi) + u(t-4\pi) + u(t-6\pi) + u(t-8\pi)\right]$$

$= \sin t$

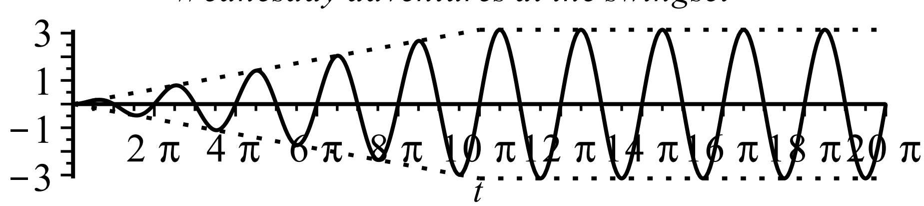
$\sin t$	$\frac{1}{s^2+1}$
$u(t-a)\,f(t-a)$	$e^{-as}\,F(s)$

$$x(t) = \begin{cases} \\ \pi \sin t\,, & t > 8\pi \end{cases}$$



```
> with(plots) :
> plot1 := plot(.1·t·sin(t), t = 0 ..10·Pi, color = black) :
  plot2 := plot(Pi·sin(t), t = 10·Pi ..20·Pi, color = black) :
  plot3 := plot(Pi, t = 10·Pi ..20·Pi, color = black, linestyle = 2) :
  plot4 := plot(−Pi, t = 10·Pi ..20·Pi, color = black, linestyle = 2) :
  plot5 := plot(.1·t, t = 0 ..10·Pi, color = black, linestyle = 2) :
  plot6 := plot(−.1·t, t = 0 ..10·Pi, color = black, linestyle = 2) :
  display({plot1, plot2, plot3, plot4, plot5, plot6}, title = `Wednesday adventures at the swingset`);
```

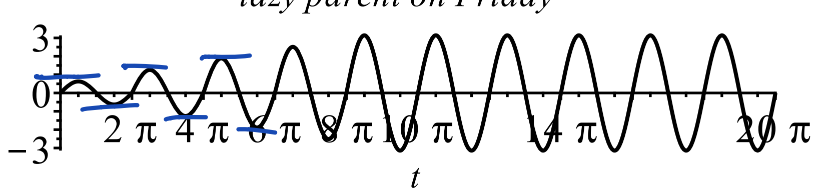
Wednesday adventures at the swingset



impulse solution: five equal impulses to get same final amplitude of π meters - Exercise 1:

```
> f := t→.2·Pi·sum(Heaviside(t − k·2·Pi)·sin(t − k·2·Pi), k = 0 ..4) :
> plot(f(t), t = 0 ..20·Pi, color = black, title = `lazy parent on Friday`);
```

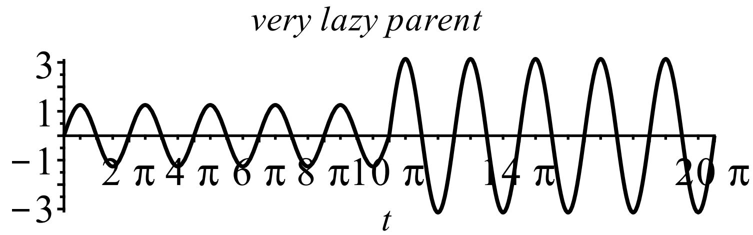
lazy parent on Friday



Or, an impulse at $t = 0$ and another one at $t = 10\,\pi$.

> $g := t \rightarrow .2 \cdot \mathrm{Pi} \cdot (2 \cdot \sin(t) + 3 \cdot \mathrm{Heaviside}(t - 10 \cdot \mathrm{Pi}) \cdot \sin(t - 10 \cdot \mathrm{Pi}))$:

> *plot*(g(t), t = 0 ..20 · Pi, *color* = *black*, *title* = \`*very lazy parent*\`);



>

• Finish Monday notes } impulse forcing }.
• Today's notes on convolution

Math 2280-001
Wed Apr 12

Convolutions and solutions to non-homogeneous physical oscillation problems (EP7.6 p. 499-501)

Consider a mechanical or electrical forced oscillation problem for $x(t)$, and the particular solution that begins at rest:

$$a\,x'' + b\,x' + c\,x = f(t)$$
$$x(0) = 0 \qquad x(0) = x_0$$
$$x'(0) = 0\,. \qquad x'(0) = v_0$$

Then in Laplace land, this equation is equivalent to

$$a\,s^2 X(s) + b\,s\,X(s) + c\,X(s) = F(s)$$
$$\Rightarrow X(s)\left(a\,s^2 + b\,s + c\right) = F(s)$$
$$\Rightarrow X(s) = F(s)\cdot\frac{1}{a\,s^2 + b\,s + c} := \underline{F(s)}\,\underline{W(s)}\,. \quad + \frac{\text{initial value terms}}{a s^2 + bs + c}$$

The inverse Laplace transform $\underline{w(t)} = \mathcal{L}^{-1}\{W(s))\}(t)$ is called the "weight function" of the given differential equation. Notice (check!) that $w(t)$ is the solution to the homogeneous DE IVP

$$a\,x'' + b\,x' + c\,x = 0$$
$$x(0) = 0$$
$$x'(0) = \cancel{1}\, 1/a$$

$\mathcal{L}$: $a\left(s^2 W(s) - s\cdot 0 - \frac{1}{a}\right) + b\left(sW(s)\right) + cW(s) = 0$

$W(s)\left(as^2 + bs + c\right) = 1$

$W(s) = \frac{1}{as^2+bs+c}$ ✓

Because of the convolution table entry

$\int_0^t f(\tau)\overset{w(t-\tau)}{g(t-\tau)}\,d\tau$	$F(s)\overset{W(s)}{G(s)}$	convolution integrals to invert Laplace transform products

sys:

$\vec{x}\,' = A\vec{x} + \vec{f}$

$\vec{x}\,' - A\vec{x} = \vec{f}$

$e^{-tA}\left(\vec{x}\,' - A\vec{x}\right) = e^{-tA}\vec{f}$

$\frac{d}{dt}\left(e^{-tA}\vec{x}\right) = e^{-tA}\vec{f}$

the solution (for ANY forcing function $f(t)$) is given by

$$\mathcal{L}^{-1}\{X(s)\}(t) = x(t) = \int_0^t f(\tau)\,w(t-\tau)\,d\tau.$$

(With non-zero initial conditions there would be homogeneous solution terms as well. In the case of damping these terms would be transient.) Notice that this says that $x(t)$ depends on the values of the forcing function $f(\tau)$ for the previous times $0 \le \tau \le t$, weighted by $w(t-\tau)$, $t \ge t-\tau \ge 0$. That the non-homogenous solutions can be constructed from the homogeneous ones via this convolution is a special case of "Duhamel's Principle", which applies to linear DE's and linear PDE's:

https://en.wikipedia.org/wiki/Duhamel%27s_principle

$\int_0^t$: $e^{-tA}\vec{x}(t) = \vec{x}_0 + \int_0^t e^{-\tau A}\vec{f}(\tau)\,d\tau$

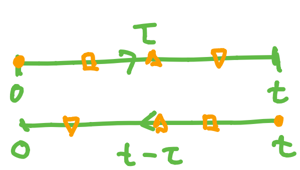
This idea generalizes to much more complicated mechanical and circuit systems, and is how engineers experiment mathematically with how proposed configurations will respond to various input forcing functions, once they figure out the weight function for their system.

$\vec{x}(t) = e^{tA}\vec{x}_0 + e^{tA}\int_0^t e^{-\tau A}\vec{f}(\tau)\,d\tau$

The mathematical justification for the general convolution table entry is at the end of today's notes.

$x(t) = \int_0^t f(\tau)\,w(t-\tau)\,d\tau$

$\vec{x}(t) = e^{tA}\vec{x}_0 + \int_0^t e^{(t-\tau)A}\vec{f}(\tau)\,d\tau$



Exercise 2. Let's play the resonance game and practice convolution integrals, first with an old friend, but then with non-sinusoidal forcing functions. We'll stick with our earlier swing, but consider various forcing periodic functions $f(t)$.

$$x''(t) + x(t) = f(t)$$
$$x(0) = 0$$
$$x'(0) = 0$$

a) $\mathcal{L}$: $X(s)(s^2+1) = F(s)$

$X(s) = F(s)\underbrace{\frac{1}{s^2+1}}_{W(s)}$

a) Find the weight function $w(t)$. $= \sin t$

b) Write down the solution formula for $x(t)$ as a convolution integral. $\boxed{x(t) = \int_0^t f(\tau)\sin(t-\tau)\,d\tau}$ $w(t) = \sin t$

c) Work out the special case of $X(s)$ when $f(t) = \cos(t)$, and verify that the ~~convolution formula~~ reproduces the answer we would've gotten from the table entry

$\mathcal{L}^{-1}$

$\frac{t}{2k}\sin(kt)$	$\frac{s}{(s^2+k^2)^2}$
$\int_0^t f(\tau)w(t-\tau)\,d\tau$	$F(s)W(s)$

illustrate convolution entry.

$$\frac{s}{(s^2+1)^2} = \frac{s}{s^2+1}\cdot\frac{1}{s^2+1}$$
$$= F(s)\,W(s)$$

table says $\mathcal{L}^{-1}(\ ) = \frac{t}{2}\sin t$

check: $f(t) = \cos t$, $w(t) = \sin t$

convolution entry says

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}(t) = \int_0^t (\cos\tau)(\sin(t-\tau))\,d\tau$$

$$= \int_0^t \cos\tau\left[\sin t\cos\tau - \cos t\sin\tau\right]d\tau$$

$$= \sin t\int_0^t \cos^2\tau\,d\tau - \cos t\int_0^t \underbrace{\cos\tau\sin\tau}_{\frac{1}{2}\sin 2\tau}\,d\tau$$

$\cos^2\tau = \frac{1+\cos 2\tau}{2}$

$$= \sin t\left(\frac{\tau}{2} + \frac{\sin 2\tau}{4}\right)\Big|_{\tau=0}^{t}$$

$$- \cos t\left[-\frac{1}{4}\cos 2\tau\right]_{\tau=0}^{t}$$

$$= (\sin t)\frac{t}{2} + \sin t\left(\frac{\sin 2t}{4}\right) - 0 - 0$$

$$- \cos t\left[-\frac{1}{4}\cos 2t + \frac{1}{4}\right] = 0$$

$'' \quad \sin t\left(\cancel{\tfrac{1}{2}\sin t\cos t}\right)$

$$+ \frac{1}{4}\cancel{\cos t}\left(\cos^2 t - \sin^2 t\right) - \frac{1}{4}\cancel{\cos t}$$

$(1 - 2\cancel{\sin^2 t})$

```
> ∫_0^t sin(τ) cos(t − τ) dτ;
  ∫_0^t cos(τ) sin(t − τ) dτ;  #convolution is commutative
```

$$\frac{1}{2}\sin(t)\,t$$

$$\frac{1}{2}\sin(t)\,t \qquad (1)$$

\>

d) Then play the resonance game on the following pages with new periodic forcing functions ...

We worked out that the solution to our DE IVP will be

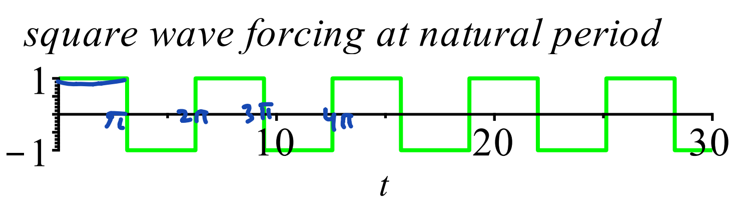
$$x(t) = \int_0^t \sin(\tau) f(t-\tau)\, d\tau \quad = w * f(t) \quad (= f * w(t)).$$

Since the unforced system has a natural angular frequency $\omega_0 = 1$, we expect resonance when the forcing function has the corresponding period of $T_0 = \frac{2\pi}{w_0} = 2\pi$. We will discover that there is the possibility for resonance if the period of f is a ***multiple*** of T_0. (Also, forcing at the natural period doesn't guarantee resonance...it depends what function you force with.)

<u>Example 1</u>) A square wave forcing function with amplitude 1 and period 2π. Let's talk about how we came up with the formula (which works until $t = 11\pi$).

> $with(plots)$:

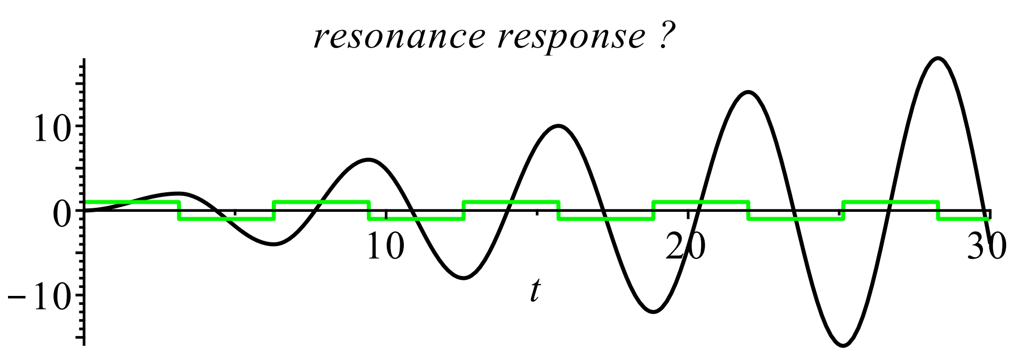
> $f1 := t \to -1 + 2 \cdot \left(\sum_{n=0}^{10} (-1)^n \cdot \text{Heaviside}(t - n \cdot \text{Pi}) \right)$:
$plot1a := plot(f1(t), t = 0..30, color = green)$:
$display(plot1a, title = \text{`square wave forcing at natural period`})$;



<u>1)</u> What's your vote? Is this square wave going to induce resonance, i.e. a response with linearly growing amplitude?

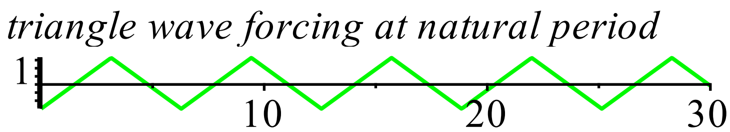
> $x1 := t \to \int_0^t \sin(\tau) \cdot f1(t-\tau)\, d\tau$:
$plot1b := plot(x1(t), t = 0..30, color = black)$:
$display(\{plot1a, plot1b\}, title = \text{`resonance response ?`})$;

✓.



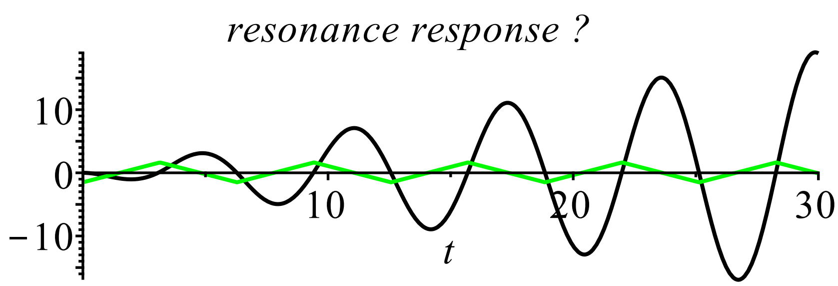
<u>Example 2</u>) A triangle wave forcing function, same period

```
> f2 := t -> ∫_0^t f1(s) ds − 1.5 :   # this antiderivative of square wave should be triangle wave
  plot2a := plot(f2(t), t = 0 ..30, color = green) :
  display(plot2a, title = `triangle wave forcing at natural period`);
```



<u>2)</u> Resonance?

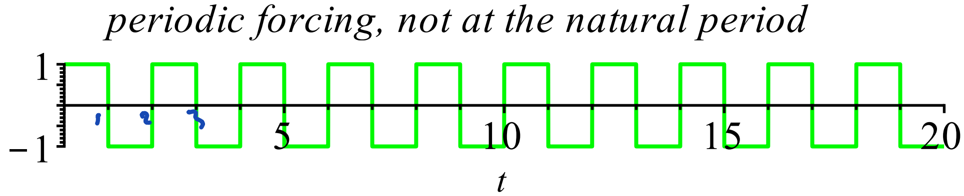
```
> x2 := t -> ∫_0^t sin(τ)·f2(t − τ) dτ :
  plot2b := plot(x2(t), t = 0 ..30, color = black) :
  display({plot2a, plot2b}, title = `resonance response ?`);
```



✓

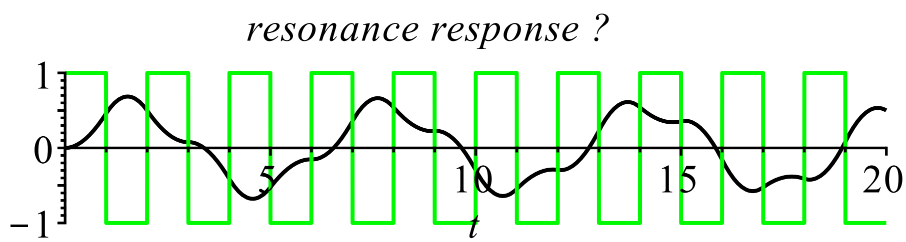
Example 3) Forcing not at the natural period, e.g. with a square wave having period $T = 2$.

> $f3 := t \to -1 + 2 \cdot \sum_{n=0}^{20} (-1)^n \cdot \mathrm{Heaviside}(t-n)$:
plot3a := *plot*(*f3*(*t*), *t* = 0 ..20, *color* = *green*) :
display(*plot3a*, *title* = \`*periodic forcing, not at the natural period*\`);



3) Resonance?

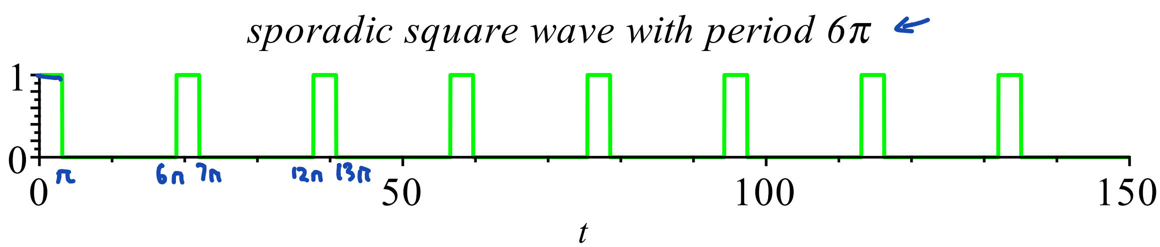
> $x3 := t \to \int_0^t \sin(\tau) \cdot f3(t-\tau)\, \mathrm{d}\tau$:
plot3b := *plot*(*x3*(*t*), *t* = 0 ..20, *color* = *black*) :
display({*plot3a*, *plot3b*}, *title* = \`*resonance response ?*\`);



Example 4) Forcing not at the natural period, e.g. with a particular wave having period $T = 6\pi$.

$$> f4 := t \to \sum_{n=0}^{10} \left(\text{Heaviside}(t - 6 \cdot n \cdot \pi) - \text{Heaviside}(t - (6 \cdot n + 1) \cdot \pi) \right) :$$

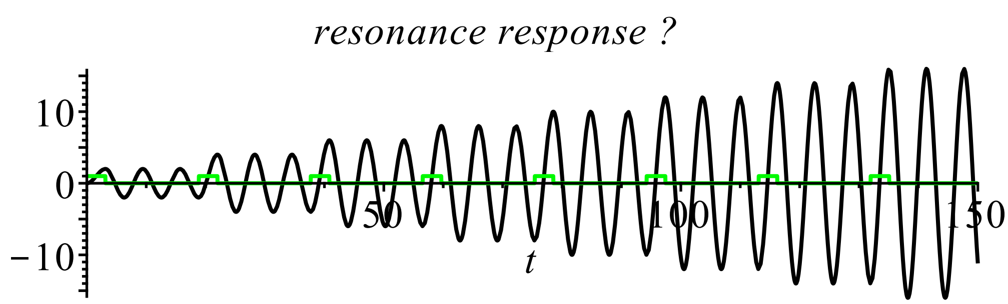
$plot4a := plot(f4(t), t = 0..150, color = green) :$
$display(plot4a, title = sporadic\ square\ wave\ with\ period\ 6\pi);$



4) Resonance?

$$> x4 := t \to \int_0^t \sin(\tau) \cdot f4(t - \tau)\, d\tau :$$

$plot4b := plot(x4(t), t = 0..150, color = black) :$
$display(\{plot4a, plot4b\}, title = `resonance\ response\ ?`);$



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Hey, what happened???? How do we need to modify our thinking if we force a system with something which is not sinusoidal, in terms of worrying about resonance? In the case that this was modeling a swing (pendulum), how is it getting pushed?

__Precise Answer:__ It turns out that any periodic function with period P is a (possibly infinite) superposition of a constant function with *cosine* and *sine* functions of periods $\left\{P, \frac{P}{2}, \frac{P}{3}, \frac{P}{4}, \ldots\right\}$. Equivalently, these functions in the superposition are

$\{1, \cos(\omega t), \sin(\omega t), \cos(2\omega t), \sin(2\omega t), \cos(3\omega t), \sin(3\omega t), \ldots\}$ with $\omega = \frac{2\cdot\pi}{P}$. This is the theory of Fourier series, which you will study in other courses, e.g. Math 3150, Partial Differential Equations. If the given periodic forcing function $f(t)$ has non-zero terms in this superposition for which $n\cdot\omega = \omega_0$ (the natural angular frequency) (equivalently $\frac{P}{n} = \frac{2\cdot\pi}{\omega_0} = T_0$), there will be resonance; otherwise, no resonance. We could already have understood some of this in Chapter 3, for example

__Exercise 3)__ The natural period of the following DE is (still) $T_0 = 2\pi$. Notice that the period of the first forcing function below is $T = 6\pi$ and that the period of the second one is $T = T_0 = 2\pi$. Yet, it is the first DE whose solutions will exhibit resonance, not the second one. Explain, using Chapter 5 superposition ideas.

__a)__

resonance! $\longrightarrow$

$$x''(t) + x(t) = \underbrace{\cos(t) + \sin\left(\frac{t}{3}\right)}.$$

$x'' + x = \cos t$ $\quad x_{p_1}(t) = t\cos(t-\alpha)$

$x'' + x = \sin\frac{t}{3}$ $\quad x_{p_2} = A\sin\left(\frac{t}{3}\right)$

$x_p = x_{p_1} + x_{p_2}$

period of forcing function

$\cos t$: 2π

$\sin\frac{t}{3}$: 6π

$\Big\}\ 6\pi$

natural period $= 2\pi$

Resonance!

__b)__

no resonance!

$$x''(t) + x(t) = \cos(2t) - 3\sin(3t).$$

$x'' + x = \cos 2t$

$x_{p_1} = A\cos 2t$

$x'' + x = -3\sin 3t$

$x_{p_2} = B\sin 3t$.

$x_p = x_{p_1} + x_{p_2}$

period.

l.c.m. $\left(\pi, \frac{2\pi}{3}\right) = 2\pi$

$= \frac{2\pi}{\omega_0}$!

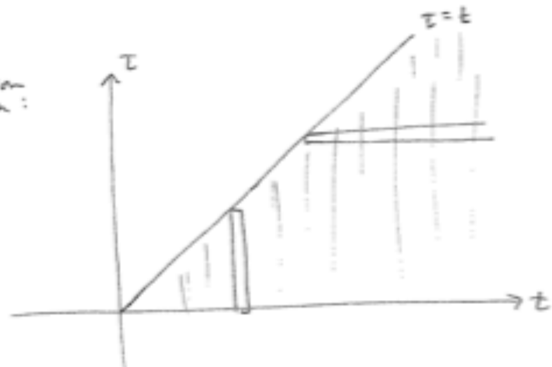
proof of convolution theorem:

(is a good review of iterated integrals)

$$\mathcal{L}\{f*g\}(s) = \int_0^\infty e^{-st}\left(\int_0^t f(\tau)\,g(t-\tau)\,d\tau\right) dt$$

$$= \int_0^\infty \int_0^t e^{-st} f(\tau)\,g(t-\tau)\,d\tau\,dt$$

integration region:



interchange limits:

$$= \int_0^\infty \int_\tau^\infty e^{-st} f(\tau)\,g(t-\tau)\,dt\,d\tau$$

$$= \int_0^\infty \int_\tau^\infty e^{-s\tau} f(\tau)\, e^{-s(t-\tau)} g(t-\tau)\,dt\,d\tau \qquad \text{(pattern recognition)}$$

$$= \int_0^\infty e^{-s\tau} f(\tau)\left[\int_\tau^\infty e^{-s(t-\tau)} g(t-\tau)\,dt\right] d\tau$$

$\tilde{t} = t-\tau$

$d\tilde{t} = dt$

$$\underbrace{\left[\int_0^\infty e^{-s\tilde{t}} g(\tilde{t})\,d\tilde{t}\right]}_{G(s)}$$

$$= G(s)\int_0^\infty e^{-s\tau} f(\tau)\,d\tau$$

$$= G(s)\,F(s) \qquad !!$$