Exercise 9) Use Laplace transforms to write down the solution to

$$x''(t) + \omega_0^2\, x(t) = \frac{F_0}{m} \sin(\omega_0 t)$$
$$x(0) = x_0$$
$$x'(0) = v_0 .$$

what phenomena do solutions to this DE illustrate (even though we're forcing with $\sin(\omega_0 t)$ rather than $\cos(\omega_0 t)$)? How would you have tried to solve this problem in Chapter 3?

$\mathcal{L}$: $\quad s^2 X(s) - s\,x_0 - v_0 + \omega_0^2 X(s) = \frac{F_0}{m}\,\frac{\omega_0}{s^2+\omega_0^2}$

$$X(s)\left(s^2+\omega_0^2\right) = \frac{F_0}{m}\,\frac{\omega_0}{s^2+\omega_0^2} + x_0 s + v_0$$

$$X(s) = \frac{F_0\omega_0}{m}\,\frac{1}{(s^2+\omega_0^2)^2} + x_0\,\frac{s}{s^2+\omega_0^2} + v_0\,\frac{1}{s^2+\omega_0^2}$$

$$x(t) = \frac{F_0\omega_0}{m}\,\frac{1}{2\omega_0^2}\left(\frac{1}{\omega_0}\sin\omega_0 t - t\cos\omega_0 t\right) + x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t$$

Exercise 10) Solve the following IVP. Use this example to recall the general partial fractions algorithm.

$$x''(t) + 4\,x(t) = 8\,t\,e^{2t}$$
$$x(0) = 0$$
$$x'(0) = 1$$

$\mathcal{L}$: $\quad s^2 X(s) - s\cdot 0 - 1 + 4X(s) = 8\,\frac{1}{(s-2)^2}$

$$X(s)(s^2+4) = \frac{8}{(s-2)^2} + 1$$

$$X(s) = \frac{8}{(s-2)^2(s^2+4)} + \frac{1}{s^2+4}$$

$$X(s) = \frac{A}{(s-2)} + \frac{B}{(s-2)^2} + \frac{Cs+D}{(s^2+4)} + \frac{1}{s^2+4}$$

part frac

$$8 = A(s-2)(s^2+4) + B(s^2+4) + (Cs+D)\underbrace{(s-2)^2}_{(s^2-4s+4)}$$

@ $s=2$: $\quad 8 = 8B \Rightarrow B = 1.$

$$\begin{aligned} 8 &= 1\,(-8A+4B+4D)\\ +0s &\quad +s\,(+4A+4C-4D)\\ +0s^2 &\quad +s^2\,(-2A+B-4C+D)\\ +0s^3 &\quad +s^3\,(A+C) \end{aligned}$$

equate coeffs

$$\begin{aligned} 8 &= -8A+4+4D\\ 0 &= +4A+4C-4D\\ 0 &= -2A+1-4C+D\\ 0 &= A+C \end{aligned}$$

$$\begin{aligned} 2 &= -2A+1+\cancel{D}\\ 0 &= +A+C-\cancel{D}\\ 0 &= -2A+1-4C+\cancel{D}\\ 0 &= A+C \end{aligned}$$

$A = -\frac{1}{2}$

$C = 1/2$

$D = 0$

$$\frac{3s^3+s}{s(s+2)^3(s^2+5)^2} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{(s+2)^2} + \frac{D}{(s+2)^3} + \frac{Es+F}{s^2+5} + \frac{Gs+H}{(s^2+5)^2}$$

$$X(s) = -\frac{1}{2}\,\frac{1}{s-2} + 1\,\frac{1}{(s-2)^2} + \frac{1}{2}\,\frac{s}{(s^2+4)} + \frac{1}{s^2+4}$$

$\mathcal{L}^{-1}$:
$$x(t) = -\frac{1}{2}e^{2t} + te^{2t} + \frac{1}{2}\cos 2t + \frac{1}{2}\sin 2t$$
$$\quad (1) \qquad (2) \qquad (3) \qquad (4)$$

Wolfram checks:

WolframAlpha computational knowledge engine.

x"(t)+4*x(t)=8*t*exp(2*t),x(0)=0, x'(0)=1

Web Apps Examples Random

Input:

$$\{x''(t) + 4\,x(t) = 8\,t\,\exp(2\,t),\ x(0) = 0,\ x'(0) = 1\}$$

Open code

ODE classification:

second-order linear ordinary differential equation

Alternate form:

$$\{x''(t) = 8\,e^{2t}\,t - 4\,x(t),\ x(0) = 0,\ x'(0) = 1\}$$

Differential equation solution: (2) (4) (3)

Approximate form | Step-by-step solution

$$x(t) = \frac{1}{2}\left(e^{2t}\,(2\,t - 1) + \sin(2\,t) + \cos(2\,t)\right)$$

(1)

WolframAlpha computational knowledge engine.

partial fractions

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Assuming "partial fractions" refers to a computation | Use as a general topic instead

rational function: 8/((s-2)^2*(s^2+4))+1

Input:

partial fractions	$\frac{8}{(s-2)^2\,(s^2+4)} + \frac{1}{s^2+4}$

Open code

Result:

Step-by-step solution

$$\frac{8}{(s-2)^2\,(s^2+4)} + \frac{1}{s^2+4} = \frac{s+2}{2\,(s^2+4)} - \frac{1}{2\,(s-2)} + \frac{1}{(s-2)^2}$$

HW
7.3.20

$$F(s) = \frac{1}{s^4 - 8s^2 + 16} = \frac{1}{(s^2-4)^2} = \frac{1}{(s+2)^2(s-2)^2}$$

$$= \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{s-2} + \frac{D}{(s-2)^2}$$

$$1 = A(s+2)(s-2)^2 + B(s-2)^2 + C(s-2)(s+2)^2 + D(s+2)^2$$

$$\vdots$$

• Last Friday example
• todays material new DE's!

Math 2280-001 Week 13 April 10-14
Mon Apr 10
7.4 - 7.5

• The following Laplace transform material is useful in systems where we turn forcing functions on and off, and when we have right hand side "forcing functions" that are more complicated than what undetermined coefficients can handle.

$f(t)$ with $\lvert f(t)\rvert \le C e^{M t}$	$F(s) := \int_0^\infty f(t) e^{-s t}\, dt$ for $s > M$	comments
$u(t-a)$ unit step function	$\frac{e^{-a s}}{s}$	for turning components on and off at $t = a$.
$f(t-a)\, u(t-a)$	$e^{-a s} F(s)$	more complicated on/off
$\delta(t-a)$	$e^{-a s}$	unit impulse/delta "function"
$\int_0^t f(\tau) g(t-\tau)\, d\tau$	$F(s) G(s)$	convolution integrals to invert Laplace transform products

The unit step function with jump at $t = 0$ is defined to be

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \ge 0 \end{cases} .$$

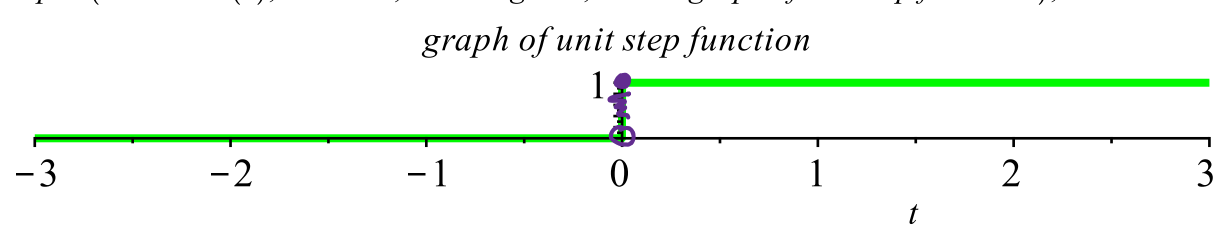
$$u(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

θ

IThis function is also called the "Heaviside" function, e.g. in Maple and Wolfram alpha. In Wolfram alpha it's also called the "theta" function. Oliver Heaviside was a an accomplished physicist in the 1800's. The name is not because the graph is heavy on one side. :-)

http://en.wikipedia.org/wiki/Oliver_Heaviside

```
> with(plots):
  plot(Heaviside(t), t=-3..3, color = green, title = `graph of unit step function`);
```



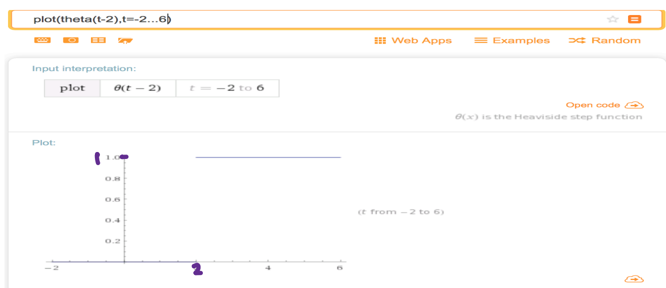
Notice that technically the vertical line should not be there - a more precise picture would have a solid point at $(0, 1)$ and a hollow circle at $(0, 0)$, for the graph of $u(t)$. In terms of Laplace transform integral definition it doesn't actually matter what we define $u(0)$ to be.

$$u(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

Then

$$u(t-a) = \begin{cases} 0, & t-a < 0; \text{ i.e. } t < a \\ 1, & t-a \geq 0; \text{ i.e. } t \geq a \end{cases}$$

and has graph that is a horizontal translation by a to the right, of the original graph, e.g. for $a = 2$:



Exercise 1) Verify the table entries

$u(t-a)$ unit step function	$\dfrac{e^{-a\,s}}{s}$	for turning components on and off at $t = a$.
$f = 1$ $f(t-a)\ u(t-a)$	$e^{-a\,s}F(s)$	more complicated on/off

$$\mathcal{L}\{f(t-a)u(t-a)\}(s) = \int_0^\infty f(t-a)\underbrace{u(t-a)}_{\substack{=0 \\ t<a} \quad \substack{=1 \\ t \geq a}} e^{-st}\,dt$$

$$= \int_0^a + \int_a^\infty$$

0

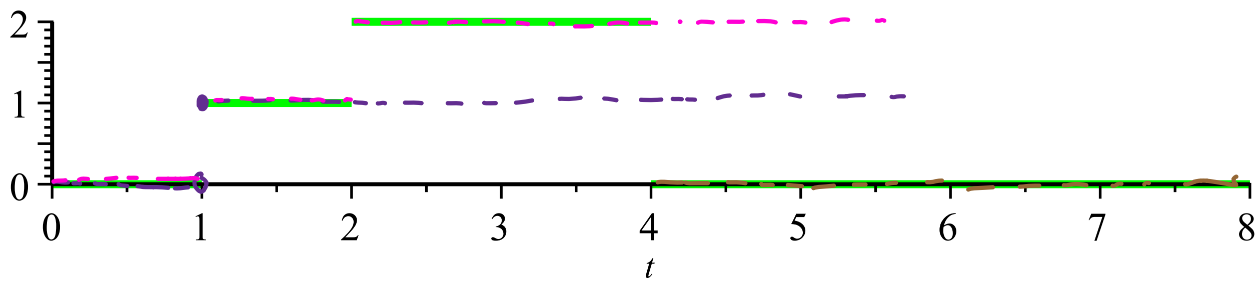
$$= \int_a^\infty f(t-a)\cdot 1\cdot e^{-st}\,dt$$

$\tilde{t} = t - a \leftarrow \tilde{t} + a$

$d\tilde{t} = dt$

$$= \int_{\substack{\tilde{t}=0 \\ t=a}}^\infty f(\tilde{t})\,e^{-s(\tilde{t}+a)}\,d\tilde{t} = e^{-as}\underbrace{\int_0^\infty f(\tilde{t})e^{-s\tilde{t}}\,d\tilde{t}}_{F(s)}$$

Exercise 2) Consider the function $f(t)$ which is zero for $t > 4$ and with the following graph. Use linearity and the unit step function entry to compute the Laplace transform $F(s)$. This should remind you of a homework problem from the assignment due tomorrow - although you're asked to find the Laplace transform of that step function directly from the definition. In your next week's homework assignment you will re-do that problem using unit step functions. (Of course, you could also check your answer in this week's homework with this method.)



$$f(t) = \begin{cases} 0 & 0 \le t < 1 \\ 1 & 1 \le t < 2 \\ 2 & 2 \le t < 4 \\ 0 & t \geq 4 \end{cases} \Longrightarrow f(t) = u(t-1) + u(t-2) - 2u(t-4)$$

$$\mathcal{L}\{f(t)\}(s) = \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} - 2\frac{e^{-4s}}{s}$$

Setup: an ~~under employed~~ dedicated mathematician/engineer/scientist
(your choice)

likes to take his/her child to the swings...

$L\theta'' + g\sin\theta = 0$

recall pendulum (linearized) eqtn, without forcing, for $\theta = \theta(t)$

$L\theta'' + g\theta = 0$

$\dashrightarrow \; x'' + g\frac{x}{L} = 0$

$\dashrightarrow \; m x'' + \frac{mg}{L} x = F_0 \cos\omega t$ ← parent forcing (?!)

$x(t) = L\sin\theta(t)$
$\approx L\theta$ for small θ

so $x'' \approx L\theta''$

$\dashrightarrow \; x'' + \frac{g}{L}x = \frac{F_0}{m}\cos\omega t$

$x'' + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$

parent pushes sinusoidally for exactly 5 cycles, and with $\frac{F_0}{m} = .2$ and then releases:

for resonance $\underline{\omega = \omega_0}$
construct swing with $L = \text{"}g\text{"} \approx 9.8$ m.
so $\omega_0^2 = 1$, $\underline{T_0 = 2\pi \approx 6.2 \text{ seconds}}$

$x'' + x = \frac{F_0}{m}\cos t$ ☺

Exercise 3a) Explain why the description above leads to the differential equation initial value problem for $x(t)$

$$x''(t) + x(t) = .2\cos(t)\left(1 - u(t - 10\pi)\right) = .2\cos t - .2\,\cancel{\cos t}\,u(t-10\pi)$$

$\cos(t-10\pi)$

$$x(0) = 0$$
$$x'(0) = 0$$

$f(t) = \cos t$

$F(s) = \frac{s}{s^2+1}$

3b) Find $x(t)$. Show that after the parent stops pushing, the child is oscillating with an amplitude of exactly π meters (in our linearized model).

$$\frac{F_0}{m} = \begin{cases} .2\cos t & 0 \le t \le 10\pi \\ 0 & t > 10\pi \end{cases}$$

$$= .2\cos t\,(1 - u(t - 10\pi))$$

$$\mathcal{L}: \quad s^2 X(s) + X(s) = .2\frac{s}{s^2+1} - .2e^{-10\pi s}\frac{s}{s^2+1}$$

$$X(s) = .2\frac{s}{(s^2+1)^2} - .2e^{-10\pi s}\underbrace{\frac{s}{(s^2+1)^2}}$$

$$x(t) = .2\tfrac{1}{2}t\sin t$$

$\mathcal{L}^{-1}$: $\frac{1}{2}t\sin t = f(t)$

$a = 10\pi$

$$-.2\tfrac{1}{2}u(t-10\pi)(t-10\pi)\sin(t-10\pi)$$

$\frac{1}{2k}t\sin kt$	$\frac{s}{(s^2+k^2)^2}$

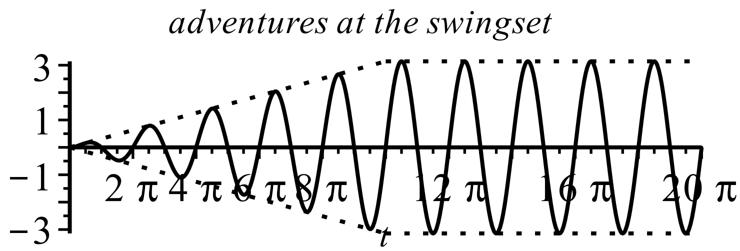
↑ this factor was missing on the blackboard.

$$x(t) = \begin{cases} .1t\sin t & 0 \le t < 10\pi \\ .1\cancel{t\sin t} - .1(\cancel{t} - 10\pi)\sin(t - 10\pi) & t \ge 10\pi \\ = \pi\sin t \;\checkmark & \end{cases}$$

$\sin(t-10\pi)$ = $\sin t$

Pictures for the swing:

```
> plot1 := plot(.1·t·sin(t), t = 0 ..10·Pi, color = black) :
  plot2 := plot(Pi·sin(t), t = 10·Pi ..20·Pi, color = black) :
  plot3 := plot(Pi, t = 10·Pi ..20·Pi, color = black, linestyle = 2) :
  plot4 := plot(-Pi, t = 10·Pi ..20·Pi, color = black, linestyle = 2) :
  plot5 := plot(.1·t, t = 0 ..10·Pi, color = black, linestyle = 2) :
  plot6 := plot(-.1·t, t = 0 ..10·Pi, color = black, linestyle = 2) :
  display({plot1, plot2, plot3, plot4, plot5, plot6}, title = `adventures at the swingset`);
```



Alternate approach via Chapter 3:

<u>step 1)</u> solve

$$x''(t) + x(t) = .2\cos(t)$$
$$x(0) = 0$$
$$x'(0) = 0$$

for $0 \le t \le 10\pi$.

<u>step 2)</u> Then solve

$$y''(t) + y(t) = 0$$
$$y(0) = x(10\pi)$$
$$y'(0) = x'(10\pi)$$

and set $x(t) = y(t - 10)$ for $t > 10$.

$f(t)$, with $\lvert f(t)\rvert \leq C e^{M t}$	$F(s) := \int_0^\infty f(t) e^{-s t}\, dt$ for $s > M$	$\downarrow$ verified
$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$	☐
1	$\frac{1}{s}$ $(s > 0)$	☐
t	$\frac{1}{s^2}$	☐
t^2	$\frac{2}{s^3}$	☐
t^n, $n \in \mathbb{N}$	$\frac{n!}{s^{n+1}}$	☐
$e^{\alpha t}$	$\frac{1}{s - \alpha}$ $(s > \Re(a))$	☐
$\cos(k t)$	$\frac{s}{s^2 + k^2}$ $(s > 0)$	☐
$\sin(k t)$	$\frac{k}{s^2 + k^2}$ $(s > 0)$	☐
$\cosh(k t)$	$\frac{s}{s^2 - k^2}$ $(s > k)$	☐
$\sinh(k t)$	$\frac{k}{s^2 - k^2}$ $(s > k)$	☐
$e^{a t}\cos(k t)$	$\frac{(s - a)}{(s - a)^2 + k^2}$ $(s > a)$	☐
$e^{a t}\sin(k t)$	$\frac{k}{(s - a)^2 + k^2}$ $(s > a)$	☐
$e^{a t} f(t)$	$F(s - a)$	☐
$u(t - a)$	$\frac{e^{-a s}}{s}$	
$f(t - a)\, u(t - a)$	$e^{-a s} F(s)$	
$\delta(t - a)$	$e^{-a s}$	
$f'(t)$	$s F(s) - f(0)$	☐
$f''(t)$	$s^2 F(s) - s f(0) - f'(0)$	☐
$f^{(n)}(t)$, $n \in \mathbb{N}$	$s^n F(s) - s^{n-1} f(0) - \ldots - f^{(n-1)}(0)$	☐

$\int_0^t f(\tau)\,d\tau$	$\dfrac{F(s)}{s}$	☐
$t\,f(t)$ $t^2\,f(t)$ $t^n\,f(t),\ n \in \mathbb{Z}$ $\dfrac{f(t)}{t}$	$-F'(s)$ $F''(s)$ $(-1)^n\,F^{(n)}(s)$ $\int_s^\infty F(\sigma)\,d\sigma$	☐ ☐ ☐
$t\cos(k\,t)$ $\dfrac{1}{2\,k}\,t\sin(k\,t)$ $\dfrac{1}{2\,k^3}\,(\sin(k\,t) - k\,t\cos(k\,t))$ $t\,e^{a\,t}$ $t^n\,e^{a\,t},\ n \in \mathbb{Z}$	$\dfrac{s^2 - k^2}{(s^2 + k^2)^2}$ $\dfrac{s}{(s^2 + k^2)^2}$ $\dfrac{1}{(s^2 + k^2)^2}$ $\dfrac{1}{(s - a)^2}$ $\dfrac{n!}{(s - a)^{n+1}}$	☐ ☐ ☐ ☐ ☐
$\int_0^t f(\tau)g(t - \tau)\,d\tau$	$F(s)G(s)$	
$f(t)$ with period p	$\dfrac{1}{1 - e^{-ps}}\int_0^p f(t)e^{-s\,t}\,dt$	

Laplace transform table