Exercise 9) Use Laplace transforms to write down the solution to
$$x''(t) + \omega_0^2 x(t) = \frac{F_0}{m}\sin(\omega_0 t)$$
$$x(0) = x_0$$
$$x'(0) = v_0 .$$

what phenomena do solutions to this DE illustrate (even though we're forcing with $\sin(\omega_0 t)$ rather than $\cos(\omega_0 t)$)? How would you have tried to solve this problem in Chapter 3?

$\mathcal{L}:$ $\quad s^2 X(s) - s x_0 - v_0 + \omega_0^2 X(s) = \frac{F_0}{m} \frac{\omega_0}{s^2 + \omega_0^2}$

$\quad\quad X(s)(s^2 + \omega_0^2) = \frac{F_0}{m} \frac{\omega_0}{s^2 + \omega_0^2} + x_0 s + v_0$

$\quad\quad X(s) = \frac{F_0 \omega_0}{m} \frac{1}{(s^2 + \omega_0^2)^2} + x_0 \frac{s}{s^2 + \omega_0^2} + v_0 \frac{1}{s^2 + \omega_0^2}$

$\quad\quad x(t) = \frac{F_0 \omega_0}{m} \frac{1}{2\omega_0^2}\left(\frac{1}{\omega_0}\sin\omega_0 t - t\cos\omega_0 t\right) + x_0 \cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t$

Exercise 10) Solve the following IVP. Use this example to recall the general partial fractions algorithm.
$$x''(t) + 4 x(t) = 8\, t\, e^{2t}$$
$$x(0) = 0$$
$$x'(0) = 1$$

$\mathcal{L}:$ $\quad s^2 X(s) - s\cdot 0 - 1 + 4 X(s) = 8\frac{1}{(s-2)^2}$

$\quad\quad X(s)(s^2 + 4) = \frac{8}{(s-2)^2} + 1$

$\quad\quad X(s) = \frac{8}{(s-2)^2(s^2+4)} + \frac{1}{s^2+4}$

$\quad\quad X(s) = \frac{A}{(s-2)} + \frac{B}{(s-2)^2} + \frac{Cs+D}{(s^2+4)} + \frac{1}{s^2+4}$

part frac

$8 = A(s-2)(s^2+4) + B(s^2+4) + (Cs+D)\underbrace{(s-2)^2}_{(s^2-4s+4)}$

@ $s=2$: $8 = 8B \Rightarrow B = 1.$

$\quad\quad\quad\quad 8 \quad\quad 1(-8A+4B+4D)$
$\quad\quad\quad +0s \quad = +s(+4A+4C-4D)$
$\quad\quad\quad +0s^2 \quad\quad +s^2(-2A+B-4C+D)$
$\quad\quad\quad +0s^3 \quad\quad +s^3(A+C)$

equate
coeffts

$8 = -8A + 4 + 4D$
$0 = +4A + 4C - 4D$
$0 = -2A + 1 - 4C + D$
$0 = A+C$

$2 = -2A + 1 + D$
$0 = +A + C - D$
$0 = -2A + 1 - 4C + D$
$0 = A+C$

$A = -\frac{1}{2}$

$\uparrow c = \frac{1}{2}$ $D = 0$

$\frac{3s^3 + s}{s(s+2)^2(s^2+5)^2}$

$= \frac{A}{s} + \frac{B}{s+2} + \frac{C}{(s+2)^2} + \frac{D}{(s+2)^3}$

$\quad + \frac{Es+F}{s^2+5} + \frac{Gs+H}{(s^2+5)^2}$

$$X(s) = -\frac{1}{2} \frac{1}{s-2} + 1 \frac{1}{(s-2)^2} + \frac{1}{2} \frac{s}{(s^2+4)} + \frac{1}{s^2+4}$$

$\mathcal{L}^{-1}$:

$$x(t) = -\frac{1}{2} e^{2t} + t e^{2t} + \frac{1}{2} \cos 2t + \frac{1}{2} \sin 2t$$

① ② ③ ④

Wolfram checks:

WolframAlpha computational knowledge engine

x"(t)+4*x(t)=8*t*exp(2*t),x(0)=0, x'(0)=1

☆ ▤

⌨ ◎ ▦ ✎   ⠿ Web Apps   ≡ Examples   ⤫ Random

Input:

$$\{x''(t) + 4x(t) = 8t \exp(2t), x(0) = 0, x'(0) = 1\}$$

Open code ☁

ODE classification:

second-order linear ordinary differential equation

Alternate form:

$$\{x''(t) = 8 e^{2t} t - 4x(t), x(0) = 0, x'(0) = 1\}$$

Differential equation solution:

② ④ ③   [Approximate form] [✎ Step-by-step solution]

$$x(t) = \frac{1}{2} \left(e^{2t} (2t - 1) + \sin(2t) + \cos(2t) \right)$$

①

☁

WolframAlpha computational knowledge engine

partial fractions

☆ ▤

⌨ ◎ ▦ ✎   ⠿ Web Apps   ≡ Examples   ⤫ Random

Assuming "partial fractions" refers to a computation | Use as a general topic instead

■ rational function: 8/((s-2)^2*(s^2+4))+1

Input:

partial fractions	$\dfrac{8}{(s-2)^2(s^2+4)} + \dfrac{1}{s^2+4}$

Open code ☁

Result: [✎ Step-by-step solution]

$$\frac{8}{(s-2)^2(s^2+4)} + \frac{1}{s^2+4} = \frac{s+2}{2(s^2+4)} - \frac{1}{2(s-2)} + \frac{1}{(s-2)^2}$$

☁

$$F(s) = \frac{1}{s^4 - 8s^2 + 16} = \frac{1}{(s^2-4)^2} = \frac{1}{(s+2)^2(s-2)^2}$$

$$= \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{s-2} + \frac{D}{(s-2)^2}$$

$$1 = A(s+2)(s-2)^2 + B(s-2)^2 + C(s-2)(s+2)^2$$
$$+ D(s+2)^2$$

$$\vdots$$

Math 2280-001 Week 13 April 10-14
Mon Apr 10
7.4 - 7.5

• The following Laplace transform material is useful in systems where we turn forcing functions on and off, and when we have right hand side "forcing functions" that are more complicated than what undetermined coefficients can handle.

$f(t)$ with $\|f(t)\| \le Ce^{M\,t}$	$F(s) := \int_0^\infty f(t)e^{-s\,t}\,dt$ for $s > M$	comments
$u(t - a)$ unit step function	$\dfrac{e^{-a\,s}}{s}$	for turning components on and off at $t = a$.
$f(t - a)\, u(t - a)$	$e^{-a\,s}F(s)$	more complicated on/off
$\delta(t - a)$	$e^{-a\,s}$	unit impulse/delta "function"
$\displaystyle\int_0^t f(\tau)g(t - \tau)\,d\tau$	$F(s)G(s)$	convolution integrals to invert Laplace transform products

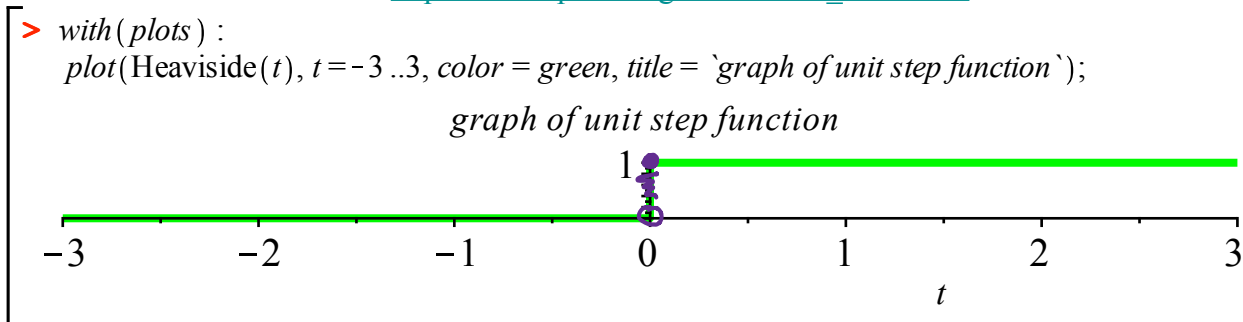The unit step function with jump at $t = 0$ is defined to be

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \ge 0 \end{cases} .$$
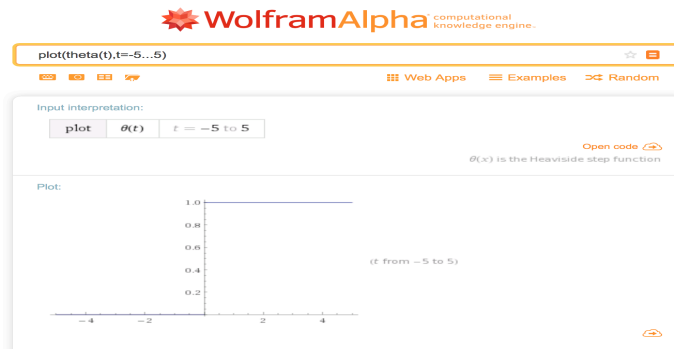
$$u(x) = \begin{cases} 0, & x < 0 \\ 1, & x \ge 0 \end{cases}$$

IThis function is also called the "Heaviside" function, e.g. in Maple and Wolfram alpha. In Wolfram alpha it's also called the "theta" function. Oliver Heaviside was a an accomplished physicist in the 1800's. The name is not because the graph is heavy on one side. :-)
                    http://en.wikipedia.org/wiki/Oliver_Heaviside

> with (plots) :
   plot(Heaviside(t), t = -3 ..3, color = green, title = `graph of unit step function`);

                    graph of unit step function

Notice that technically the vertical line should not be there - a more precise picture would have a solid point at $(0, 1)$ and a hollow circle at $(0, 0)$, for the graph of $u(t)$. In terms of Laplace transform integral definition it doesn't actually matter what we define $u(0)$ to be.

$$u(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

Then

$$u(t-a) = \begin{cases} 0, & t-a < 0; \ i.e. \ t < a \\ 1, & t-a \geq 0; \ i.e. \ t \geq a \end{cases}$$

and has graph that is a horizontal translation by a to the right, of the original graph, e.g. for $a = 2$:

<u>Exercise 1)</u> Verify the table entries

$u(t-a)$ unit step function	$\dfrac{e^{-as}}{s}$	for turning components on and off at $t = a$.
$f(t-a)\, u(t-a)$	$e^{-as} F(s)$	more complicated on/off

$f = 1$

$$\mathcal{L}\{f(t-a)u(t-a)\}(s) = \int_0^\infty f(t-a)\underbrace{u(t-a)}_{\substack{= \\ \text{"}0\text{" "}1\text{"} \\ t<a \quad t \geq a}} e^{-st}\, dt$$

$$= \int_0^a + \int_a^\infty$$

$$0$$

$$= \int_a^\infty f(t-a)\cdot 1 \cdot e^{-st}\, dt$$

$$\tilde{t} = t - a \quad \leftarrow \quad \tilde{t} + a$$
$$d\tilde{t} = dt$$

$$= \int_{\tilde{t}=0}^\infty f(\tilde{t})\, e^{-s(\tilde{t}+a)}\, d\tilde{t} = e^{-as} \underbrace{\int_0^\infty f(\tilde{t})\, e^{-s\tilde{t}}\, d\tilde{t}}_{F(s)}$$

$$t = a$$

Exercise 2) Consider the function $f(t)$ which is zero for $t > 4$ and with the following graph. Use linearity and the unit step function entry to compute the Laplace transform $F(s)$. This should remind you of a homework problem from the assignment due tomorrow - although you're asked to find the Laplace transform of that step function directly from the definition. In your next week's homework assignment you will re-do that problem using unit step functions. (Of course, you could also check your answer in this week's homework with this method.)
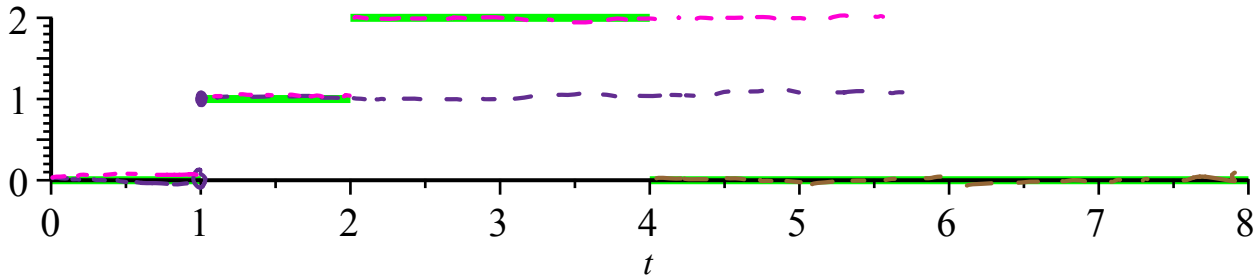
$$f(t) = \begin{cases} 0 & 0 \leq t < 1 \\ 1 & 1 \leq t < 2 \\ 2 & 2 \leq t < 4 \\ 0 & t \geq 4 \end{cases} \implies f(t) = u(t-1) + u(t-2) - 2u(t-4)$$

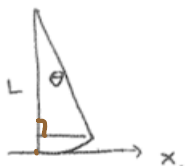$$\mathcal{L}\{f(t)\}(s) = \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} - 2\frac{e^{-4s}}{s}$$

Setup: an ~~under employed~~ *dedicated* mathematician/engineer/scientist
(your choice)

likes to take his/her child to the swings...

$L\theta'' + g\sin\theta = 0$

recall pendulum (linearized) eqtn, without forcing, fn $\theta = \theta(t)$

$L\theta'' + g\theta = 0$

$\longrightarrow x'' + g\frac{x}{L} = 0$

$\longrightarrow mx'' + \frac{mg}{L}x = F_0\cos\omega t \quad\leftarrow$ parent forcing (?!)

$x(t) = L\sin\theta(t)$
$\cong L\theta$ for small θ

so $x'' \cong L\theta''$

$\longrightarrow x'' + \frac{g}{L}x = \frac{F_0}{m}\cos\omega t$

$x'' + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$

parent pushes sinusoidally for
exactly 5 cycles, and
with $\frac{F_0}{m} = .2$ and then releases:

for resonance $\omega = \omega_0$
construct swing with $L = g$, $'' \approx 9.8$ m.
so $\omega_0^2 = 1$, $T_0 = 2\pi \approx 6.2$ seconds

$x'' + x = \frac{F_0}{m}\cos t$

Exercise 3a) Explain why the description above leads to the differential equation initial value problem for $x(t)$

$$x''(t) + x(t) = .2\cos(t)(1 - u(t - 10\pi)) = .2\cos t - .2\cos t\, u(t-10\pi)$$
$$\cos(t-10\pi)$$
$$f(t) = \cos t$$
$$F(s) = \frac{s}{s^2+1}$$
$$x(0) = 0$$
$$x'(0) = 0$$

3b) Find $x(t)$. Show that after the parent stops pushing, the child is oscillating with an amplitude of exactly π meters (in our linearized model).

$$\frac{F_0}{m} = \begin{cases} .2\cos t & 0 \le t \le 10\pi \\ 0 & t > 10\pi \end{cases}$$

$$= .2\cos t\,(1 - u(t - 10\pi))$$

$\mathcal{L}:\quad s^2 X(s) + X(s) = .2\frac{s}{s^2+1} - .2 e^{-10\pi s}\frac{s}{s^2+1}$

$X(s) = .2\frac{s}{(s^2+1)^2} - .2 e^{-10\pi s}\frac{s}{(s^2+1)^2}$

$x(t) = .2\frac{1}{2}t\sin t$

$\downarrow \mathcal{L}^{-1}: \frac{1}{2}t\sin t = f(t)$

$a = 10\pi$

$-.2\frac{1}{2}u(t-10\pi)(t-10\pi)\sin(t-10\pi)$

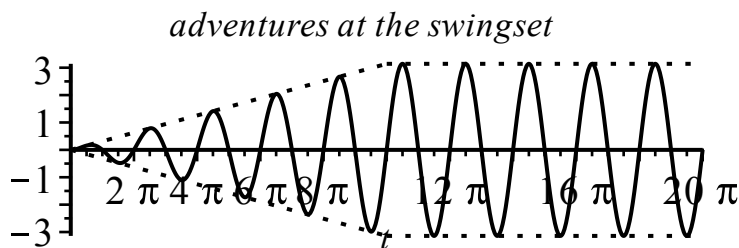$\frac{1}{2h}t\sin ht \quad \frac{s}{(s^2+h^2)^2}$

↑

this factor
was missing on
the blackboard.

$x(t) = \begin{cases} .1t\sin t & 0 \le t < 10\pi \\ .1t\sin t - .1(t-10\pi)\sin(t - 10\pi) & t \ge 10\pi \\ \qquad\qquad\qquad\qquad\; \sin t \\ = \pi\sin t \quad\checkmark \end{cases}$

Pictures for the swing:

> $plot1 := plot(.1 \cdot t \cdot \sin(t), t = 0 ..10 \cdot Pi, color = black) :$
> $plot2 := plot(Pi \cdot \sin(t), t = 10 \cdot Pi ..20 \cdot Pi, color = black) :$
> $plot3 := plot(Pi, t = 10 \cdot Pi ..20 \cdot Pi, color = black, linestyle = 2) :$
> $plot4 := plot(-Pi, t = 10 \cdot Pi ..20 \cdot Pi, color = black, linestyle = 2) :$
> $plot5 := plot(.1 \cdot t, t = 0 ..10 \cdot Pi, color = black, linestyle = 2) :$
> $plot6 := plot(-.1 \cdot t, t = 0 ..10 \cdot Pi, color = black, linestyle = 2) :$
> $display(\{plot1, plot2, plot3, plot4, plot5, plot6\}, title = \text{`adventures at the swingset`});$

adventures at the swingset

Alternate approach via Chapter 3:

<u>step 1)</u> solve

$$x''(t) + x(t) = .2 \cos(t)$$
$$x(0) = 0$$
$$x'(0) = 0$$

for $0 \leq t \leq 10 \pi$.

<u>step 2)</u> Then solve

$$y''(t) + y(t) = 0$$
$$y(0) = x(10 \pi)$$
$$y'(0) = x'(10 \pi)$$

and set $x(t) = y(t - 10)$ for $t > 10$.

$f(t),$ with $|f(t)| \le Ce^{Mt}$	$F(s) := \int_0^\infty f(t)e^{-st}\,dt$ for $s > M$	$\downarrow$ verified
$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$	☐
1	$\dfrac{1}{s}$ $\quad (s > 0)$	☐
t	$\dfrac{1}{s^2}$	☐
t^2	$\dfrac{2}{s^3}$	☐
$t^n,\ n \in \mathbb{N}$	$\dfrac{n!}{s^{n+1}}$	☐
$e^{\alpha t}$	$\dfrac{1}{s-\alpha}$ $\quad (s > \Re(a))$	☐
$\cos(kt)$	$\dfrac{s}{s^2+k^2}$ $\quad (s > 0)$	☐
$\sin(kt)$	$\dfrac{k}{s^2+k^2}$ $\quad (s > 0)$	☐
$\cosh(kt)$	$\dfrac{s}{s^2-k^2}$ $\quad (s > k)$	☐
		☐
$\sinh(kt)$	$\dfrac{k}{s^2-k^2}$ $\quad (s > k)$	☐
$e^{at}\cos(kt)$		☐
$e^{at}\sin(kt)$	$\dfrac{(s-a)}{(s-a)^2+k^2}$ $\quad (s > a)$	
	$\dfrac{k}{(s-a)^2+k^2}$ $\quad (s > a)$	☐
$e^{at}f(t)$	$F(s-a)$	☐
$u(t-a)$	$\dfrac{e^{-as}}{s}$	
$f(t-a)\,u(t-a)$	$e^{-as}F(s)$	
$\delta(t-a)$	e^{-as}	
$f'(t)$	$s\,F(s) - f(0)$	☐
$f''(t)$	$s^2 F(s) - s\,f(0) - f'(0)$	☐
$f^{(n)}(t),\ n \in \mathbb{N}$	$s^n F(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$	☐

$\displaystyle\int_0^t f(\tau)\,d\tau$	$\dfrac{F(s)}{s}$	☐
$t\,f(t)$ $t^2\,f(t)$ $t^n\,f(t),\ n \in \mathbb{Z}$ $\dfrac{f(t)}{t}$	$-F'(s)$ $F''(s)$ $(-1)^n\,F^{(n)}(s)$ $\displaystyle\int_s^\infty F(\sigma)\,d\sigma$	☐ ☐ ☐ ☐
$t\,\cos(k\,t)$ $\dfrac{1}{2\,k}\,t\,\sin(k\,t)$ $\dfrac{1}{2\,k^3}\,(\sin(k\,t) - k\,t\,\cos(k\,t))$ $t\,e^{a\,t}$ $t^n\,e^{a\,t},\ n \in \mathbb{Z}$	$\dfrac{s^2 - k^2}{(s^2 + k^2)^2}$ $\dfrac{s}{(s^2 + k^2)^2}$ $\dfrac{1}{(s^2 + k^2)^2}$ $\dfrac{1}{(s - a)^2}$ $\dfrac{n!}{(s - a)^{n+1}}$	☐ ☐ ☐ ☐ ☐
$\displaystyle\int_0^t f(\tau)g(t - \tau)\,d\tau$	$F(s)G(s)$	
$f(t)$ with period p	$\dfrac{1}{1 - e^{-ps}}\displaystyle\int_0^p f(t)e^{-s\,t}\,dt$	

Laplace transform table

<u>EP 7.6</u> impulse functions and the δ operator.

Consider a force $f(t)$ acting on an object for only on a very short time interval $a \le t \le a + \varepsilon$, for example as when a bat hits a ball. This <u>impulse</u> p of the force is defined to be the integral

$$p := \int_a^{a + \varepsilon} f(t) \, dt$$

and it measures the <u>net change in momentum</u> of the object since by Newton's second law

$$m \, v'(t) = f(t)$$

$$\Rightarrow \int_a^{a + \varepsilon} m \, v'(t) \, dt = \int_a^{a + \varepsilon} f(t) \, dt = p$$

$$\Rightarrow m \, v(t) \, \Big]_{t = a}^{a + \varepsilon} = p \, .$$

Since the impulse p only depends on the integral of $f(t)$, and since the exact form of f is unlikely to be known in any case, the easiest model is to replace f with a constant force having the same total impulse, i.e. to set

$$f = p \, d_{a, \, \varepsilon}(t)$$

where $d_{a, \, \varepsilon}(t)$ is the <u>unit impulse</u> function given by

$$d_{a, \, \varepsilon}(t) = \begin{cases} 0, & t < a \\ \dfrac{1}{\varepsilon}, & a \le t < a + \varepsilon \\ 0, & t \ge a + \varepsilon \end{cases} .$$
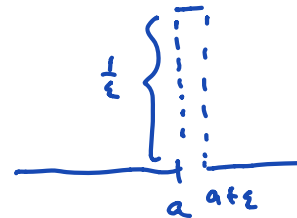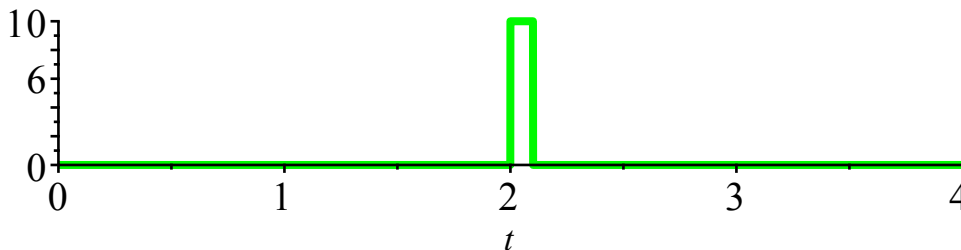
Notice that

$$\int_a^{a + \varepsilon} d_{a, \, \varepsilon}(t) \, dt = \int_a^{a + \varepsilon} \frac{1}{\varepsilon} \, dt = 1 \, .$$

Here's a graph of $d_{2, \, .1}(t)$, for example:

Since the unit impulse function is a linear combination of unit step functions, we could solve differential equations with impulse functions so-constructed. As far as Laplace transform goes, it's even easier to take the limit as $\varepsilon \to 0$ for the Laplace transforms $\mathcal{L}\{d_{a, \, \varepsilon}(t)\}(s)$, and this effectively models impulses on very short time scales.

$$d_{a, \, \varepsilon}(t) = \frac{1}{\varepsilon}[u(t - a) - u(t - (a + \varepsilon))]$$

$$\frac{d}{d\varepsilon} \, 1 - e^{-\varepsilon s} = -e^{-\varepsilon s}(-s)$$
$$= se^{-\varepsilon s}$$

$$\Rightarrow \mathcal{L}\{d_{a,\varepsilon}(t)\}(s) = \frac{1}{\varepsilon}\left(\frac{e^{-a\,s}}{s} - \frac{e^{-(a+\varepsilon)s}}{s}\right)$$

$$= e^{-a\,s}\left(\frac{1-e^{-\varepsilon s}}{\varepsilon\,s}\right).$$

@ $\varepsilon \to 0$ $\left(\frac{1-e^{-\varepsilon s}}{\varepsilon s}\right) = \frac{0}{0}$

In Laplace land we can use L'Hopital's rule (in the variable ε) to take the limit as $\varepsilon \to 0$:

$$\lim_{\varepsilon \to 0} e^{-a\,s}\left(\frac{1-e^{-\varepsilon s}}{\varepsilon\,s}\right) = e^{-a\,s}\lim_{\varepsilon \to 0}\left(\frac{s\,e^{-\varepsilon s}}{s}\right) = e^{-a\,s}.$$

L'Hopital

The result in time t space is not really a function but we call it the "delta function" $\delta(t-a)$ anyways, and visualize it as a function that is zero everywhere except at $t = a$, and that it is infinite at $t = a$ in such a way that its integral over any open interval containing a equals one. As explained in EP7.6, the delta "function" can be thought of in a rigorous way as a linear transformation, not as a function. It can also be thought of as the derivative of the unit step function $u(t-a)$, and this is consistent with the Laplace table entries for derivatives of functions. In any case, this leads to the very useful Laplace transform table entry

$\delta(t-a)$ unit impulse function delta function	$e^{-a\,s}$	for impulse forcing

Exercise 4) Revisit the swing. In this case the parent is providing an impulse each time the child passes through equilibrium position after completing a cycle.

$$x''(t) + x(t) = .2\,\pi\left[\delta(t) + \delta(t - 2\,\pi) + \delta(t - 4\,\pi) + \delta(t - 6\,\pi) + \delta(t - 8\,\pi)\right]$$
$$x(0) = 0$$
$$x'(0) = 0.$$

$x(0)$
$x'(0) = 0$

$\mathcal{L}:\quad X(s)(s^2 + 1) = .2\pi\left[1 + e^{-2\pi s} + e^{-4\pi s} + e^{-6\pi s} + e^{-8\pi s}\right]$

$$X(s) = .2\pi\left[\frac{1}{s^2+1} + e^{-2\pi s}\cdot\frac{1}{s^2+1} + \cdots + e^{-8\pi s}\cdot\frac{1}{s^2+1}\right]$$

$f(t)$

$$X(t) = .2\pi\left[\sin t + u(t-2\pi)\sin(t-2\pi) + u(t-4\pi)\sin(t-4\pi) + \cdots\right]$$

$= \sin t$ $\sin t$

$$x(t) = .2\pi\left[\sin t\right]\left\{1 + u(t-2\pi) + u(t-4\pi) + u(t-6\pi) + u(t-8\pi)\right\}$$

$\sin t \;\Big|\; \dfrac{1}{s^2+1}$

$u(t-a) f(t-a) \;\Big|\; e^{-as} F(s)$

$$x(t) = \begin{cases} \qquad\qquad \\ \pi\sin t, \quad t > 8\pi \end{cases}$$

```
> with(plots) :
> plot1 := plot(.1·t·sin(t), t = 0 ..10·Pi, color = black) :
  plot2 := plot(Pi·sin(t), t = 10·Pi ..20·Pi, color = black) :
  plot3 := plot(Pi, t = 10·Pi ..20·Pi, color = black, linestyle = 2) :
  plot4 := plot(−Pi, t = 10·Pi ..20·Pi, color = black, linestyle = 2) :
  plot5 := plot(.1·t, t = 0 ..10·Pi, color = black, linestyle = 2) :
  plot6 := plot(−.1·t, t = 0 ..10·Pi, color = black, linestyle = 2) :
  display({plot1, plot2, plot3, plot4, plot5, plot6}, title = `Wednesday adventures at the swingset`);
```
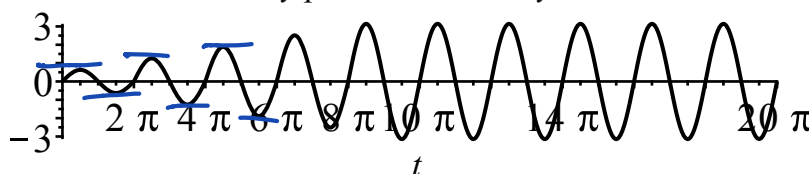
Wednesday adventures at the swingset
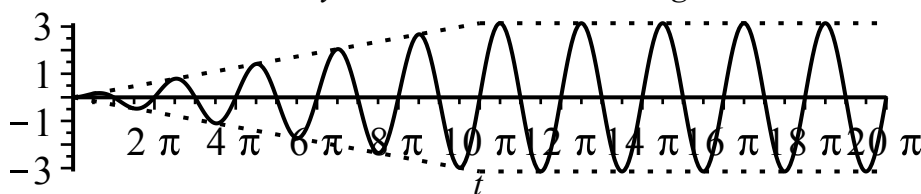
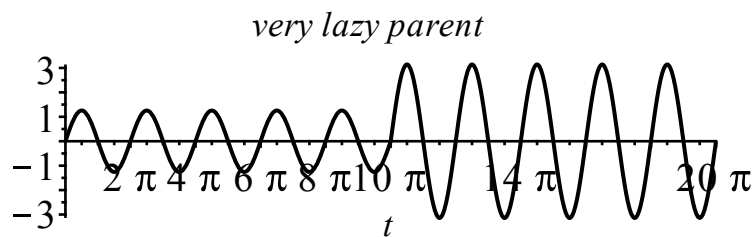impulse solution: five equal impulses to get same final amplitude of π meters - Exercise 1:
```
> f := t→.2·Pi·sum(Heaviside(t − k·2·Pi)·sin(t − k·2·Pi), k = 0 ..4) :
> plot(f(t), t = 0 ..20·Pi, color = black, title = `lazy parent on Friday`);
```

lazy parent on Friday

Or, an impulse at $t = 0$ and another one at $t = 10\,\pi$.

> $g := t \rightarrow .2 \cdot \text{Pi} \cdot (2 \cdot \sin(t) + 3 \cdot \text{Heaviside}(t - 10 \cdot \text{Pi}) \cdot \sin(t - 10 \cdot \text{Pi})) :$
> $plot(g(t), t = 0 .. 20 \cdot \text{Pi}, color = black, title = \text{`very lazy parent`});$

very lazy parent

>

• Finish Monday notes }
  impulse forcing
• Today's notes on convolution

Convolutions and solutions to non-homogeneous physical oscillation problems (EP7.6 p. 499-501)

Consider a mechanical or electrical forced oscillation problem for $x(t)$, and the particular solution that begins at rest:
$$a x'' + b x' + c x = f(t)$$
$$x(0) = 0$$
$$x'(0) = 0.$$

$x(0) = x_0$
$x'(0) = v_0$

Then in Laplace land, this equation is equivalent to
$$a s^2 X(s) + b s X(s) + c X(s) = F(s)$$
$$\Rightarrow X(s)(a s^2 + b s + c) = F(s)$$
$$\Rightarrow X(s) = F(s) \cdot \frac{1}{a s^2 + b s + c} := F(s) W(s).$$

$+ \dfrac{\text{inital value terms}}{a s^2 + b s + c}$

The inverse Laplace transform $w(t) = \mathcal{L}^{-1}\{W(s))\}(t)$ is called the "weight function" of the given differential equation. Notice (check!) that $w(t)$ is the solution to the homogeneous DE IVP
$$a x'' + b x' + c x = 0$$
$$x(0) = 0$$
$$x'(0) = \frac{1}{a}$$

$\mathcal{L}:$ $a(s^2 W(s) - s \cdot 0 - \frac{1}{a})$
$+ b(s W(s)) + c W(s) = 0$
$W(s)(a s^2 + b s + c) = 1$
$W(s) = \frac{1}{a s^2 + b s + c}$ ✓

Because of the convolution table entry

$w(t-\tau)$ $\int_0^t f(\tau) g(t-\tau)\, d\tau$	$W(s)$ $F(s)G(s)$	convolution integrals to invert Laplace transform products

(sys)

$\vec{x}' = A\vec{x} + \vec{f}$
$\vec{x}' - A\vec{x} = \vec{f}$
$e^{-tA}(\vec{x}' - A\vec{x}) = e^{-tA}\vec{f}$
$\frac{d}{dt}(e^{-tA}\vec{x}) = e^{-tA}\vec{f}$

the solution (for ANY forcing function $f(t)$) is given by
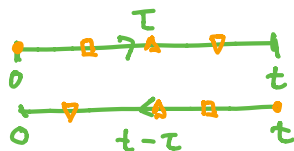$$\mathcal{L}^{-1}\{X(s)\}(t) = x(t) = \int_0^t f(\tau) w(t-\tau)\, d\tau.$$

(With non-zero initial conditions there would be homogeneous solution terms as well. In the case of damping these terms would be transient.) Notice that this says that $x(t)$ depends on the values of the forcing function $f(\tau)$ for the previous times $0 \le \tau \le t$, weighted by $w(t-\tau), t \ge t - \tau \ge 0$. That the non-homogenous solutions can be constructed from the homogeneous ones via this convolution is a special case of "Duhamel's Principle", which applies to linear DE's and linear PDE's:
https://en.wikipedia.org/wiki/Duhamel%27s_principle

$\int_0^t:$ $e^{-tA}\vec{x}(t) = \vec{x}_0$
$+ \int_0^t e^{-\tau A}\vec{f}(\tau)\, d\tau$

This idea generalizes to much more complicated mechanical and circuit systems, and is how engineers experiment mathematically with how proposed configurations will respond to various input forcing functions, once they figure out the weight function for their system.

$\vec{x}(t) = e^{tA}\vec{x}_0$

$+ e^{tA}\int_0^t e^{-\tau A}\vec{f}(\tau)\, d\tau$

The mathematical justification for the general convolution table entry is at the end of today's notes.

$x(t) = \int_0^t f(\tau) w(t-\tau)\, d\tau$

$\vec{x}(t) = e^{tA}\vec{x}_0$
$+ \int_0^t e^{(t-\tau)A} \vec{f}(\tau)\, d\tau$

Exercise 2. Let's play the resonance game and practice convolution integrals, first with an old friend, but then with non-sinusoidal forcing functions. We'll stick with our earlier swing, but consider various forcing periodic functions $f(t)$.

$$x''(t) + x(t) = f(t)$$
$$x(0) = 0$$
$$x'(0) = 0$$

a) $\mathcal{L}: X(s)(s^2+1) = F(s)$

$$X(s) = F(s) \underbrace{\frac{1}{s^2+1}}_{W(s)}$$

a) Find the weight function $w(t) = \sin t$

b) Write down the solution formula for $x(t)$ as a convolution integral. $x(t) = \int_0^t f(\tau)\sin(t-\tau)\,d\tau$ $w(t) = \sin t$

c) Work out the special case of $X(s)$ when $f(t) = \cos(t)$, and verify that the ~~convolution formula~~ reproduces the answer we would've gotten from the table entry

$\mathcal{L}^{-1}$ illustrate convolution entry.

$\dfrac{t}{2k}\sin(kt)$	$\dfrac{s}{(s^2+k^2)^2}$
$\int_0^t f(\tau)w(t-\tau)\,d\tau$	$F(s)W(s)$

$$\frac{s}{(s^2+1)^2} = \frac{s}{s^2+1} \cdot \frac{1}{s^2+1}$$
$$= F(s)\,W(s)$$

table says $\mathcal{L}^{-1}(\;) = \frac{t}{2}\sin t$

check: $f(t) = \cos t$, $w(t) = \sin t$
convolution entry says

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}(t) = \int_0^t (\cos\tau)(\sin(t-\tau))\,d\tau$$

$$= \int_0^t \cos\tau\left[\sin t \cos\tau - \cos t \sin\tau\right]d\tau$$

$$= \sin t \int_0^t \cos^2\tau\,d\tau - \cos t \int_0^t \cos\tau\sin\tau\,d\tau$$

$$\underbrace{= \sin t\left(\frac{t}{2} + \frac{\sin 2\tau}{4}\right)t}_{\tau=0} \quad \underbrace{}_{\frac{1+\cos 2t}{2}} \quad \underbrace{}_{\frac{1}{2}\sin 2\tau}$$

$$- \cos t \left[-\frac{1}{4}\cos 2\tau\right]_{\tau=0}^{t}$$

> $\int_0^t \sin(\tau)\cos(t-\tau)\,d\tau;$

$\int_0^t \cos(\tau)\sin(t-\tau)\,d\tau;$ #convolution is commutative

$$= (\sin t)\frac{t}{2} + \sin t\left(\frac{\sin 2t}{4}\right) - 0 - 0$$
$$- \cos t\left[-\frac{1}{4}\cos 2t + \frac{1}{4}\right]$$

$$= 0$$

$\dfrac{1}{2}\sin(t)\,t$

$\dfrac{1}{2}\sin(t)\,t$

$\sin t\left(\frac{1}{2}\sin t \cos t\right)$ (1)

$+\frac{1}{4}\cos t\left(\cos^2 t - \sin^2 t\right) - \frac{1}{4}\cos t$

d) Then play the resonance game on the following pages with new periodic forcing functions ... $(1 - 2s\cdot nt)$

We worked out that the solution to our DE IVP will be

$$x(t) = \int_0^t \sin(\tau) f(t - \tau) \, d\tau$$ $= w * f(t)$ $\left(= f * w(t) \right)$.

Since the unforced system has a natural angular frequency $\omega_0 = 1$, we expect resonance when the forcing

function has the corresponding period of $T_0 = \dfrac{2\pi}{w_0} = 2\pi$. We will discover that there is the possibility

for resonance if the period of f is a ***multiple*** of T_0. (Also, forcing at the natural period doesn't guarantee resonance...it depends what function you force with.)
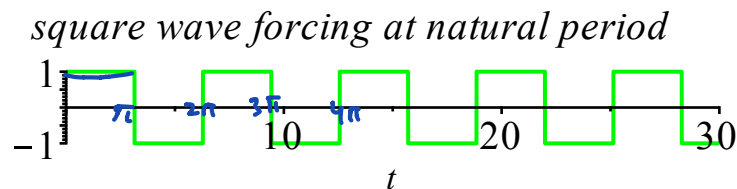
Example 1) A square wave forcing function with amplitude 1 and period 2π. Let's talk about how we came up with the formula (which works until $t = 11\pi$).

```
> with(plots) :
```

```
> f1 := t → -1 + 2 · ( Σ_{n=0}^{10} (-1)^n · Heaviside(t − n·Pi) ) :
```
```
  plot1a := plot( f1(t), t = 0 ..30, color = green) :
  display( plot1a, title = `square wave forcing at natural period`);
```


square wave forcing at natural period
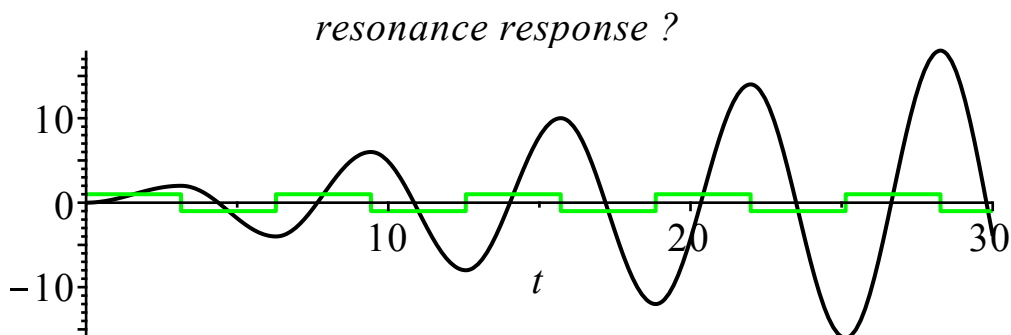
1) What's your vote? Is this square wave going to induce resonance, i.e. a response with linearly growing amplitude?

```
> x1 := t → ∫_0^t sin(τ) · f1(t − τ) dτ :
```
```
  plot1b := plot(x1(t), t = 0 ..30, color = black) :
  display( {plot1a, plot1b}, title = `resonance response ?`);
```

✓.

resonance response ?

Example 2) A triangle wave forcing function, same period

> $f2 := t \rightarrow \int_0^t f1(s) \, ds - 1.5 :$ # *this antiderivative of square wave should be triangle wave*

   $plot2a := plot(f2(t), t = 0..30, color = green) :$
   $display(plot2a, title = \text{`triangle wave forcing at natural period`});$

triangle wave forcing at natural period

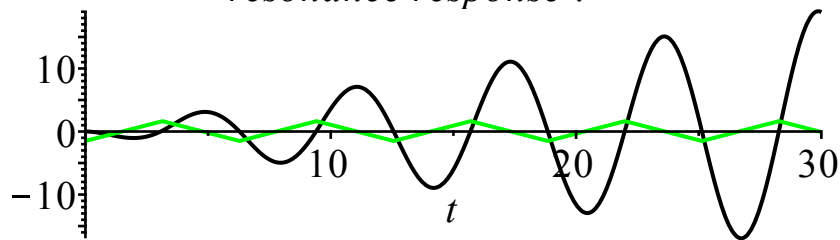**2)** Resonance?

> $x2 := t \rightarrow \int_0^t \sin(\tau) \cdot f2(t - \tau) \, d\tau :$
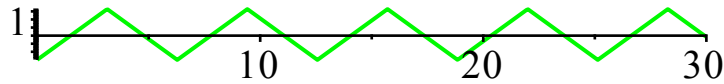
   $plot2b := plot(x2(t), t = 0..30, color = black) :$
   $display(\{plot2a, plot2b\}, title = \text{`resonance response ?`});$
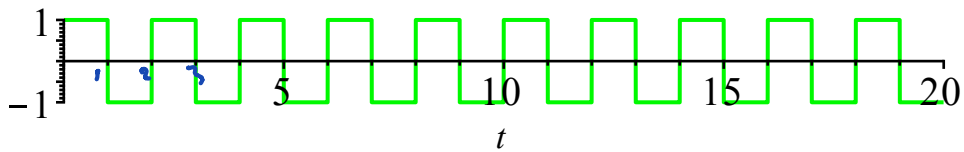

resonance response ?

Example 3) Forcing not at the natural period, e.g. with a square wave having period $T = 2$.

$$> f3 := t \rightarrow -1 + 2 \cdot \sum_{n=0}^{20} (-1)^n \cdot \text{Heaviside}(t - n) :$$

$$plot3a := plot(f3(t), t = 0..20, color = green) :$$
$$display(plot3a, title = `periodic\ forcing,\ not\ at\ the\ natural\ period`);$$
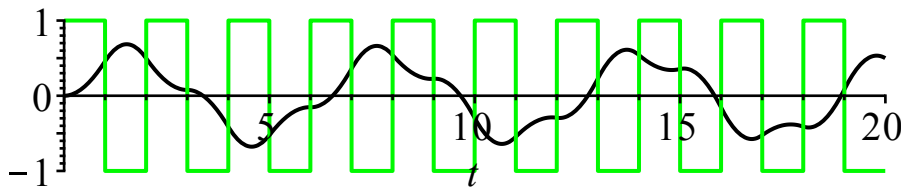
periodic forcing, not at the natural period

3) Resonance?

$$> x3 := t \rightarrow \int_0^t \sin(\tau) \cdot f3(t - \tau)\ d\tau :$$

$$plot3b := plot(x3(t), t = 0..20, color = black) :$$
$$display(\{plot3a, plot3b\}, title = `resonance\ response\ ?`);$$

resonance response ?

Example 4) Forcing not at the natural period, e.g. with a particular wave having period $T = 6\pi$.

> $f4 := t \rightarrow \displaystyle\sum_{n=0}^{10} \left(\text{Heaviside}(t - 6 \cdot n \cdot \pi) - \text{Heaviside}(t - (6 \cdot n + 1) \cdot \pi) \right) :$

$plot4a := plot(f4(t), t = 0 .. 150, color = green) :$
$display(plot4a, title = \textit{sporadic square wave with period } 6\pi);$

sporadic square wave with period 6π ⬅

4) Resonance?

> $x4 := t \rightarrow \displaystyle\int_0^t \sin(\tau) \cdot f4(t - \tau) \, d\tau :$
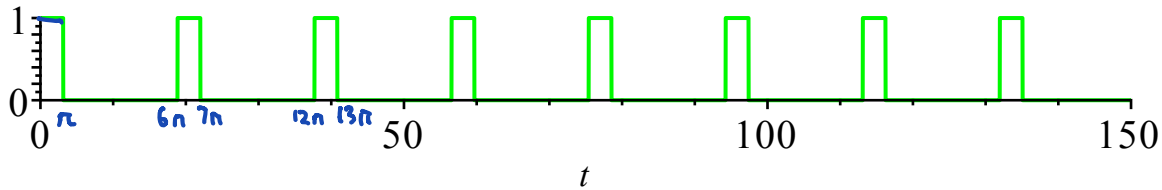
$plot4b := plot(x4(t), t = 0 .. 150, color = black) :$
$display(\{plot4a, plot4b\}, title = \text{`resonance response ?`});$

resonance response ?

>
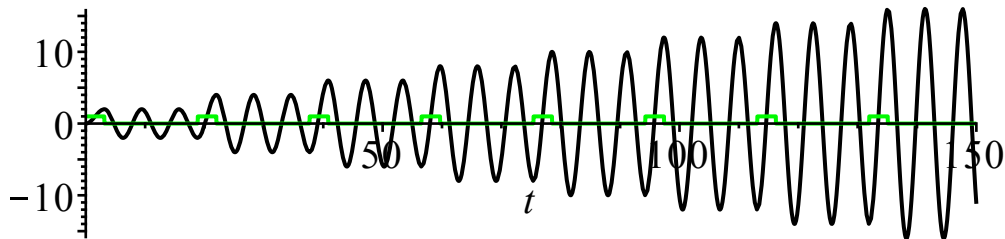
Hey, what happened???? How do we need to modify our thinking if we force a system with something which is not sinusoidal, in terms of worrying about resonance? In the case that this was modeling a swing (pendulum), how is it getting pushed?

Precise Answer: It turns out that <u>any periodic function with period P</u> is a (possibly infinite) superposition of a constant function with *cosine* and *sine* functions of periods $\left\{P, \dfrac{P}{2}, \dfrac{P}{3}, \dfrac{P}{4}, ...\right\}$. Equivalently, these functions in the superposition are

$\left\{1, \cos(\omega t), \sin(\omega t), \cos(2\omega t), \sin(2\omega t), \cos(3\omega t), \sin(3\omega t), ...\right\}$ with $\omega = \dfrac{2\cdot\pi}{P}$. This is the theory of Fourier series, which you will study in other courses, e.g. Math 3150, Partial Differential Equations. If the given periodic forcing function $f(t)$ has non-zero terms in this superposition for which $n\cdot\omega = \omega_0$ (the natural angular frequency) (equivalently $\dfrac{P}{n} = \dfrac{2\cdot\pi}{\omega_0} = T_0$), there will be resonance; otherwise, no resonance. We could already have understood some of this in Chapter 3, for example

<u>Exercise 3)</u> The natural period of the following DE is (still) $T_0 = 2\pi$. Notice that the period of the first forcing function below is $T = 6\pi$ and that the period of the second one is $T = T_0 = 2\pi$. Yet, it is the first DE whose solutions will exhibit resonance, not the second one. Explain, using Chapter 5 superposition ideas.

a)

resonance! $\longrightarrow$ $x''(t) + x(t) = \underline{\cos(t) + \sin\left(\dfrac{t}{3}\right)}$.

$x'' + x = \cos t$ $\quad x_{p_1}(t) = t\,C\cos(t-\alpha)$

$x'' + x = \sin\dfrac{t}{3}$ $\quad x_{p_2} = A\sin\left(\dfrac{t}{3}\right)$

$x_p = x_{p_1} + x_{p_2}$

b)

no resonance! $\quad x''(t) + x(t) = \cos(2t) + 3\sin(3t)$.

$x'' + x = \cos 2t$
$x_{p_1} = A\cos 2t$

$x'' + x = -3\sin 3t$
$x_{p_2} = B\sin 3t$.

$x_p = x_{p_1} + x_{p_2}$

period of forcing function
$\cos t : 2\pi$
$\sin\dfrac{t}{3} : 6\pi$ $\Big\}$ 6π
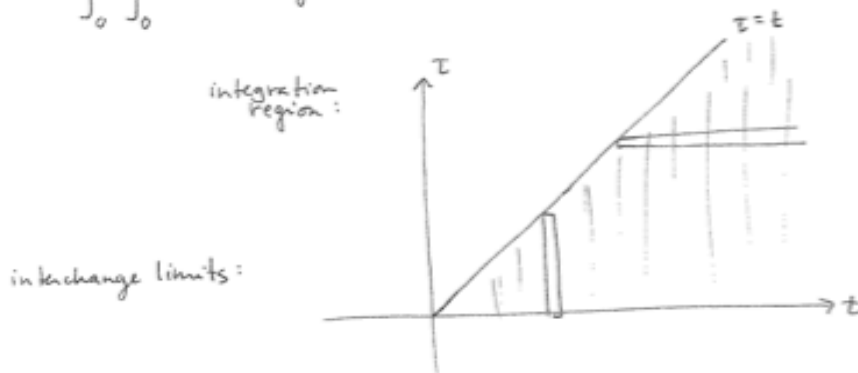
natural period $= 2\pi$
Resonance!

period.
l.c.m. $\left(\pi, \dfrac{2\pi}{3}\right) = 2\pi$
$= \dfrac{2\pi}{\omega_0}$!

proof of convolution theorem:
  (is a good review of iterated integrals)

$$\mathcal{L}\{f*g\}(s) = \int_0^\infty e^{-st}\left(\int_0^t f(\tau)\,g(t-\tau)\,d\tau\right)dt$$

$$= \int_0^\infty \int_0^t e^{-st}\,f(\tau)\,g(t-\tau)\,d\tau\,dt$$

integration region:

interchange limits:

$$= \int_0^\infty \int_\tau^\infty e^{-st}\,f(\tau)\,g(t-\tau)\,dt\,d\tau$$

$$= \int_0^\infty \int_\tau^\infty e^{-s\tau}\,f(\tau)\,e^{-s(t-\tau)}\,g(t-\tau)\,dt\,d\tau \qquad \text{(pattern recognition)}$$

$$= \int_0^\infty e^{-s\tau}\,f(\tau)\left[\int_\tau^\infty e^{-s(t-\tau)}\,g(t-\tau)\,dt\right]d\tau$$

$$\tilde{t} = t-\tau$$
$$d\tilde{t} = dt$$

$$\left[\int_0^\infty e^{-s\tilde{t}}\,g(\tilde{t})\,d\tilde{t}\right]$$

$$\underbrace{\phantom{\int_0^\infty e^{-s\tilde{t}}\,g(\tilde{t})\,d\tilde{t}}}_{G(s)}$$

$$= G(s)\int_0^\infty e^{-st}\,f(\tau)\,d\tau$$

$$= G(s)\,F(s) \qquad \text{!!}$$

Math 2280-001
Fri Apr 14

Chapter 9 Fourier Series and applications to differential equations (and partial differential equations)
9.1-9.2 Fourier series definition and convergence.

The idea of Fourier series is related to the linear algebra concepts of dot product, norm, and projection.
We'll review this connection after the definition of Fourier series:

Let $f: [-\pi, \pi] \to \mathbb{R}$ be a piecewise continuous function, or equivalently, extend to $f: \mathbb{R} \to \mathbb{R}$ as a
2π-periodic function.
Example one could consider the 2π-periodic extension of $f(t) = |t|$, initially defined on the t-interval
$[-\pi, \pi]$, to all of $\mathbb{R}$. Its graph is the so-called "tent function", $tent(t)$
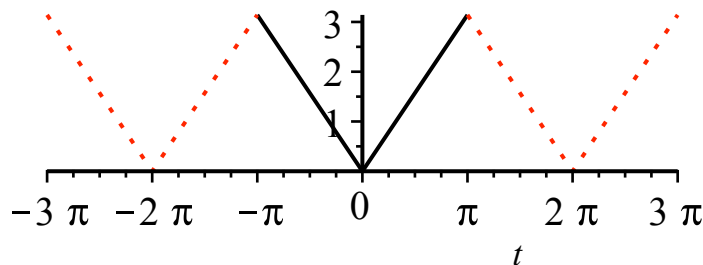
The Fourier coefficents of a 2π-periodic function f are computed via the definitions

$$a_0 := \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\, dt$$

$$a_n := \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\cos(n\,t)\, dt, \, n \in \mathbb{N}$$

$$b_n := \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\sin(n\,t)\, dt, \, n \in \mathbb{N}$$

And the Fourier series for f is given by

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\,t) + \sum_{n=1}^{\infty} b_n \sin(n\,t).$$

The idea is that the partial sums of the Fourier series of f should actually converge to f. The reasons why
this should be true combine linear algebra ideas related to orthonormal basis vectors and projection, with
analysis ideas related to convergence. Let's do an example to illustrate the magic, before discussing (parts
of) why the convergence actually happens.

Exercise 1 Consider the even function $f(t) = |t|$ on the interval $-\pi \le t \le \pi$, extended to be the 2π-periodic "tent function" $tent(t)$ of page 1. Find the Fourier coefficients a_0, a_n, b_n and the Fourier series for $tent$. The answer is below, along with a graph of partial sum of the Fourier series.

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\cos nt \, dt$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\sin nt \, dt = \frac{1}{\pi} \int_{-\pi}^{\pi} |t| \sin(nt) \, dt$$

$$\underbrace{}_{\text{odd fcn}}$$

$$= 0 \, !$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} |t| \, dt$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |t| \cos nt \, dt$$

$$= \frac{2}{\pi} \int_0^\pi t \, dt$$

$$\underbrace{\left[\tfrac{1}{2}t^2\right]_0^\pi}$$

$$\underbrace{\tfrac{\pi^2}{2}}$$

$$a_0 = \pi$$

$$= \frac{2}{\pi} \int_0^\pi \overset{=u}{t} \underset{=dv}{\cos nt} \, dt \qquad v = +\frac{\sin(nt)}{n}$$

$$a_n = \frac{2}{\pi}\left[t\left(\frac{\sin nt}{n}\right)\Big|_{t=0}^{\cancel{}} \; - \; \int_0^\pi \frac{\sin(nt)}{n}\, dt \right]$$

$$a_n = \frac{2}{\pi}\left[-\; -\frac{\cos nt}{n^2}\right]_0^\pi = \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{2}{\pi n^2}\left[\cos(n\pi) - 1\right] = \begin{cases} 0 & \text{never} \\ -\frac{4}{\pi n^2} & n \text{ odd} \end{cases}$$

solution: $tent \sim \dfrac{\pi}{2} - \dfrac{4}{\pi}\displaystyle\sum_{n\ \text{odd}} \dfrac{1}{n^2}\cos(n t)$

$\uparrow$
$\frac{a_0}{2}$

> $f1 := t \to \dfrac{\pi}{2} - \dfrac{4}{\pi}\displaystyle\sum_{j=0}^{4}\dfrac{1}{(2\cdot j + 1)^2}\cos((2 j + 1)\cdot t):$

> $plot(f1(t), t = -10 .. 10, color = black);$
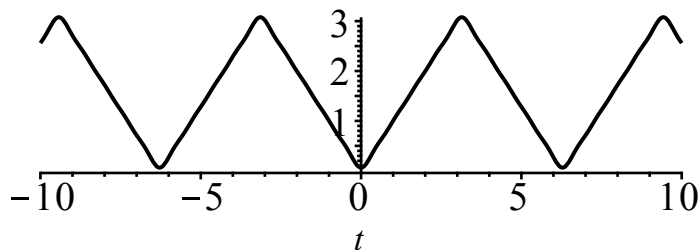
Using technology to compute Fourier coefficients:

> $f := t \rightarrow |t|;$

$$f := t \rightarrow |t| \tag{2}$$

> $a0 := \dfrac{1}{\pi} \cdot \displaystyle\int_{-\pi}^{\pi} f(t) \, dt;$

$assume(n, integer);$ # this will let Maple attempt to evaluate the integrals

$a := n \rightarrow \dfrac{1}{\pi} \cdot \displaystyle\int_{-\pi}^{\pi} f(t) \cdot \cos(n \cdot t) \, dt :$

$b := n \rightarrow \dfrac{1}{\pi} \cdot \displaystyle\int_{-\pi}^{\pi} f(t) \cdot \sin(n \cdot t) \, dt :$

$a(n);$
$b(n);$

$$a0 := \pi$$

$$\frac{2\left((-1)^{n\sim} - 1\right)}{\pi \, n\sim^2}$$

$$0 \tag{3}$$

I believe Wolfram alpha can do this too

>

So what's going on?

Recall the ideas of dot product, angle, orthonormal basis and projection onto subspaces, in $\mathbb{R}^N$, from linear algebra:

For $\underline{x}, \underline{y} \in \mathbb{R}^N$, the dot product $\underline{x} \cdot \underline{y} := \sum_{k=1}^{N} x_k y_k$ satisfies for all vectors $\underline{x}, \underline{y}, \underline{z} \in \mathbb{R}^N$ and scalars $s \in \mathbb{R}$:

a) $\underline{x} \cdot \underline{x} \geq 0$ and $= 0$ if and only if $\underline{x} = \underline{0}$

b) $\underline{x} \cdot \underline{y} = \underline{y} \cdot \underline{x}$

c) $\underline{x} \cdot (\underline{y} + \underline{z}) = \underline{x} \cdot \underline{y} + \underline{x} \cdot \underline{z}$

d) $(s\,\underline{x}) \cdot \underline{y} = s(\underline{x} \cdot \underline{y}) = \underline{x} \cdot (s\,\underline{y})$

From these four properties one can define the *norm* or magnitude of a vector by
$$\|\underline{x}\| = \sqrt{\underline{x} \cdot \underline{x}}$$
and the distance bewteen two vectors $\underline{x}, \underline{y}$ by
$$dist(\underline{x}, \underline{y}) := \|\underline{x} - \underline{y}\|.$$

One can check with algebra that the *Cauchy-Schwarz inequality* holds:
$$|\underline{x} \cdot \underline{y}| \leq \|\underline{x}\|\ \|\underline{y}\|,$$

with equality if and only if $\underline{x}, \underline{y}$ are scalar multiples of each other. One consequence of the Cauchy-Schwarz inequality is the *triangle inequality*
$$\|\underline{x} + \underline{y}\| \leq \|\underline{x}\| + \|\underline{y}\|,$$

with equality if and only if $\underline{x}, \underline{y}$ are non-negative scalar multiples of each other. Equivalently, in terms of Euclidean distance,

$$dist(\underline{x}, \underline{z}) \leq dist(\underline{x}, \underline{y}) + dist(\underline{y}, \underline{z}).$$

Another consequence of the Cauchy-Schwarz inequality is that one can define the angle θ between $\underline{x}, \underline{y}$ via
$$\cos(\theta) := \frac{\underline{x} \cdot \underline{y}}{\|\underline{x}\|\ \|\underline{y}\|},$$
for $0 \leq \theta \leq \pi$, because $-1 \leq \dfrac{\underline{x} \cdot \underline{y}}{\|\underline{x}\|\ \|\underline{y}\|} \leq 1$ holds so that θ exists. In particular two vectors $\underline{x}, \underline{y}$ are perpendicular, or *orthogonal* if and only if
$$\underline{x} \cdot \underline{y} = 0.$$

If one has a $n-dimensional$ subspace $W \subseteq \mathbb{R}^N$ an *orthonormal basis* $\{\underline{u}_1, \underline{u}_2, \ldots \underline{u}_n\}$ for W is a collection of unit vectors (<u>normalized</u> to length 1), which are also mutually <u>orthogonal</u>. (One can find such bases via the Gram-Schmidt algorithm.) For such an ortho-normal basis the nearest point projection of a vector $\underline{x} \in \mathbb{R}^N$ onto W is given by

$$proj_W\underline{x} = (\underline{x} \cdot \underline{u}_1)\underline{u}_1 + (\underline{x} \cdot \underline{u}_2)\underline{u}_2 + \ldots + (\underline{x} \cdot \underline{u}_n)\underline{u}_n = \sum_{k=1}^{n}(\underline{x} \cdot \underline{u}_k)\underline{u}_k.$$

For any $\underline{x}$ (already) in W, $proj_W\underline{x} = \underline{x}$.

The entire algebraic/geometric development on the previous page just depended on the four algebraic properties **a,b,c,d** for the dot product. So it can be generalized:

<u>Definition</u> Let V is any real-scalar vector space. we call V an <u>inner product space</u> if there is an <u>inner product</u> $\langle f, g \rangle$
for which the inner product satisfies $\forall f, g, h \in V$ and scalars $s \in \mathbb{R}$:
a) $\langle f, f \rangle \geq 0.\ \langle f, f \rangle = 0.$
b) $\langle f, g \rangle = \langle g, f \rangle.$
c) $\langle f, g + h \rangle = \langle f, g \rangle + \langle f, h \rangle$
d) $\langle (s f), g \rangle = s \langle f, g \rangle = \langle f, s\, g \rangle.$

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\, g(t)\, dt$$

In this case one can define $\| f \| = \sqrt{\langle f, f \rangle}$, $dist(f, g) = \| f - g \|$; prove the Cauchy-Schwarz inequality and the triangle inequalities; define angles between vectors, and in particular, orthogonality between vectors; find ortho-normal bases $\{u_1, u_2, \ldots u_n\}$ for finite-dimensional subspaces W, and prove that for any $f \in V$ the nearest element in W to f is given by

$$proj_W f = \langle f, u_1 \rangle u_1 + \langle f, u_2 \rangle u_2 + \ldots + \langle f, u_n \rangle u_n = \sum_{k=1}^{n} <f, u_k > u_k.$$

<u>Theorem</u> Let $V = \{ f: \mathbb{R} \to \mathbb{R}\ s.t.\ f$ is piecewise continuous and 2π-periodic$\}$. Define

$$\langle f, g \rangle := \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) g(t)\, dt.$$

1) Then $V, \langle\,,\,\rangle$ is an inner product space.

2) Let $V_N := span\left\{ \dfrac{1}{\sqrt{2}}, \cos(t), \cos(2t), \ldots, \cos(Nt), \sin(t), \sin(2t), \ldots \sin(Nt) \right\}$. Then the
$2N+1$ functions listed in this collection are an orthonormal basis for the $(2N+1)$ dimensional subspace V_N. In particular, for any $f \in V$ the nearest function in V_N to f is given by

$$proj_{V_N} f = <f, \underbrace{\frac{1}{\sqrt{2}}>}_{\langle f, 1 \rangle \frac{1}{2}} \frac{1}{\sqrt{2}} + \sum_{n=1}^{N} <f, \cos(nt)> \cos(nt) + \sum_{n=1}^{N} <f, \sin(nt)> \sin(nt)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(nt) + \sum_{n=1}^{N} b_n \sin(nt)$$

where a_0, a_n, b_n are the Fourier coefficients defined on page 1.

<u>Exercise 2)</u> Check that $\left\{ \dfrac{1}{\sqrt{2}}, \cos(t), \cos(2\,t), ..., \cos(N\,t), \sin(t), \sin(2\,t), ... \sin(N\,t) \right\}$ are

orthonormal with respect to the inner product

$$\langle f, g \rangle := \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)g(t)\ \mathrm{d}t$$

Hint:

$$\cos((m+k)t) = \cos(m\,t)\cos(k\,t) + \sin(m\,t)\sin(k\,t)$$
$$\sin((m+k)t) = \sin(m\,t)\cos(k\,t) + \cos(m\,t)\sin(k\,t)$$

so

$$\cos(m\,t)\cos(k\,t) = \frac{1}{2}\left(\cos((m+k)\,t) + \cos((m-k)t)\right) \quad \text{(even if } m=k) \ ✗$$

$$\sin(m\,t)\sin(k\,t) = \frac{1}{2}\left(\cos((m-k)\,t) - \cos((m+k)t)\right) \quad \text{(even if } m=k)$$

$$\cos(m\,t)\sin(k\,t) = \frac{1}{2}\left(\sin((m+k)\,t) + \sin((-m+k)t)\right)$$

e.g. $\langle \cos mt, \cos kt \rangle = 0 \qquad k \neq m.$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}(\cos mt)(\cos kt)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{\cos(m+k)t + \cos(m-k)t}{2}\right)dt$$

$$= \frac{1}{\pi}\left[\frac{\sin((m+k)t)}{2(m+k)} + \frac{\sin((m-k)t)}{2(m-k)}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\cdot 0 = 0$$
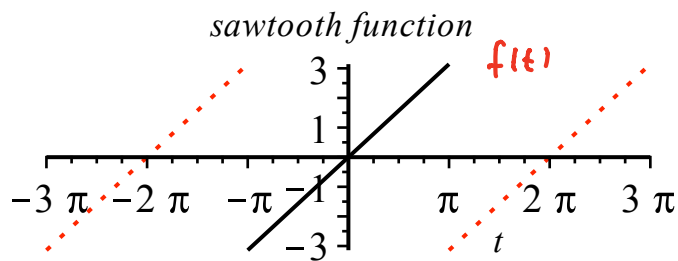
$m=k:$ is $\|\cos mt\| = 1$?

$$\|\cdot\|^2 = \frac{1}{\pi}\int_{-\pi}^{\pi}(\cos mt)^2\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1 + \cos(2mt)}{2}\,dt$$

$$= \frac{1}{\pi}\left[\frac{t}{2} + \frac{\sin(2mt)}{4m}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\pi + 0\right] = 1 \ ✓$$

Exercise 3) Consider the 2π − periodic odd function $saw(t)$ define by extending
$$f(t) = t, \quad -\pi < t \le \pi$$
as a 2π − periodic function.

sawtooth function

Find the Fourier series for $saw(t)$. Hint: you noticed that for the *even* tent function in Exercise 1 the *sine* Fourier coefficients were all zero. Which ones will be zero for any *odd* function? Why?

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{t\cos nt}_{odd\ fcn}\, dt = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{t\sin nt}_{even\ fcn}\, dt = \frac{2}{\pi}\int_{0}^{\pi} \underset{\substack{\uparrow\\ u}}{t}\ \underbrace{\sin(nt)}_{dv}\, dt =$$
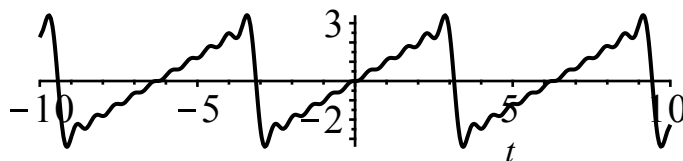
$$du = dt \qquad v = -\frac{\cos nt}{n}$$

$$= \frac{2}{\pi}\left[\ t\left(-\frac{\cos nt}{n}\right)\Big|_{0}^{\pi} - \int_{0}^{\pi} -\frac{\cos nt}{n}\, dt\ \right] \nearrow 0.$$

$$= \frac{2}{\pi}\left[\frac{\pi}{n}\left(-(-1)^{n}\right)\right] = \frac{2}{n}(-1)^{n+1}$$

solution: $saw \sim 2\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{n}\sin(nt)$

> $f2 := t \rightarrow 2\cdot\sum_{n=1}^{10}\dfrac{(-1)^{n+1}}{n}\cdot\sin(n\cdot t)$:

> $plot(f2(t), t = -10 .. 10, color = black);$

Convergence Theorems (These require some careful mathematical analysis to prove - they are often discussed in Math 5210, for example.)

Theorem 1 Let $f: \mathbb{R} \to \mathbb{R}$ be 2π-periodic and piecewise continuous. Let
$$f_N = proj_{V_N} f = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(nt) + \sum_{n=1}^{N} b_n \sin(nt)$$
be the Fourier series truncated at N. Then
$$\lim_{n \to \infty} \|f - f_N\| = \lim_{n \to \infty} \left[\frac{1}{\pi} \int_{-\pi}^{\pi} (f(t) - f_N(t))^2 \, dt \right]^{\frac{1}{2}} = 0.$$
In other words, the distance between f_N and f converges to zero, where we are using the distance function that we get from the inner product,
$$dist(f, g) = \|f - g\| = \sqrt{\langle f - g, f - g \rangle} = \left[\frac{1}{\pi} \int_{-\pi}^{\pi} (f(t) - g(t))^2 \, dt \right]^{\frac{1}{2}}.$$

Theorem 2 If f is as in Theorem 1, and is (also) piecewise differentiable with at most jump discontinuities, then

(i) for any t_0 such that f is differentiable at t_0
$$\lim_{N \to \infty} f_N(t_0) = f(t_0) \quad \text{(pointwise convergence)}.$$
(ii) for any t_0 where f is not differentiable (but is either continuous or has a jump discontinuity), then
$$\lim_{N \to \infty} f_N(t_0) = \frac{1}{2} \left(f_-(t_0) + f_+(t_0) \right)$$
where
$$f_-(t_0) = \lim_{t \to t_0^-} f(t), \quad f_+(t_0) = \lim_{t \to t_0^+} f(t)$$

Examples:
1) The truncated Fourier series for the tent function, $tent_N(t)$ converge to $tent(t)$ for all t. In fact, it can be shown that the convergence is <u>uniform</u>, i.e. $\forall\, \varepsilon > 0 \, \exists\, N$ s.t. $n \geq N \Rightarrow |tent(t) - tent_n(t)| < \varepsilon$ <u>for all t at once.</u>
2) The truncated Fourier series for the sawtooth function, $saw_N(t)$ converge to $saw(t)$ for all

$t \neq \pi + 2k\pi$, $k \in \mathbb{Z}$ (i.e. everywhere except at the jump points). At these jump points the Fourier series converges to the average of the left and right hand limits of saw, which is 0. (In fact, each partial sum evaluates to 0 at those points.) The convergence at the other t values is pointwise, but not uniform, as the convergence takes longer nearer the jump points.)

Exercise 4) We can derive "magic" summation formulas using Fourier series. (See your homework for some more.) From Theorem 2 we know that the Fourier series for $tent(t)$ converges for all t. In particular

$$0 = tent(0) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{n^2} \cos(n \cdot 0).$$

4a) Deduce

$$\frac{1}{1} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \sum_{n \text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{8}.$$

4b) Verify and use

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n \text{ odd}} \frac{1}{n^2} + \sum_{n \text{ even}} \frac{1}{n^2}$$

$$= \sum_{n \text{ odd}} \frac{1}{n^2} + \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

to show

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Differentiating Fourier Series:

Theorem 3 Let f be 2π-periodic, piecewise differentiable and continuous, and with f' piecewise continuous. Let f have Fourier series

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\,t) + \sum_{n=1}^{\infty} b_n \sin(n\,t).$$

Then f' has the Fourier series you'd expect by differentiating term by term:

$$f' \sim \sum_{n=1}^{\infty} -n\,a_n \sin(n\,t) + \sum_{n=1}^{\infty} n\,b_n \cos(n\,t)$$

proof: Let f' have Fourier series

$$f' \sim \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n\,t) + \sum_{n=1}^{\infty} B_n \sin(n\,t).$$

Then

$$A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f'(t) \cos(n\,t)\, dt,\ n \in \mathbb{N}.$$

Integrate by parts with $u = \cos(n\,t)$, $dv = f'(t)dt$, $du = -n\,\sin(n\,t)dt$, $v = f(t)$:

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f'(t)\cos(n\,t)\, dt = \frac{1}{\pi} f(t)(-n)\sin(nt) \big]_{-\pi}^{\pi} - \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)(-n)\sin(n\,t)\, dt$$

$$= 0 + \frac{n}{\pi} \int_{-\pi}^{\pi} f(t)\sin(n\,t)\, dt = n\,b_n.$$

Similarly, $A_0 = 0$, $B_n = -n\,a_n$.

□

Remark: This is analogous to what happened with Laplace transform. In that case, the transform of the derivative multiplied the transform of the original function by s (and there were correction terms for the initial values). In this case the tranformed variables are the a_n, b_n which depend on n. And the Fourier series "transform" of the derivative of a function multiplies these coefficients by n (and permutes them).

<u>Exercise 5a</u> Use the differentiation theorem and the Fourier series for *tent*(t) to find the Fourier series for the square wave, *square*(t), which is the 2π- periodic extension of

$$f(t) = \begin{cases} -1 & -\pi < t < 0 \\ 1 & 0 < t < \pi \end{cases}$$
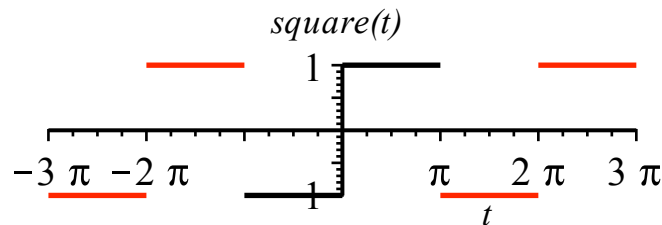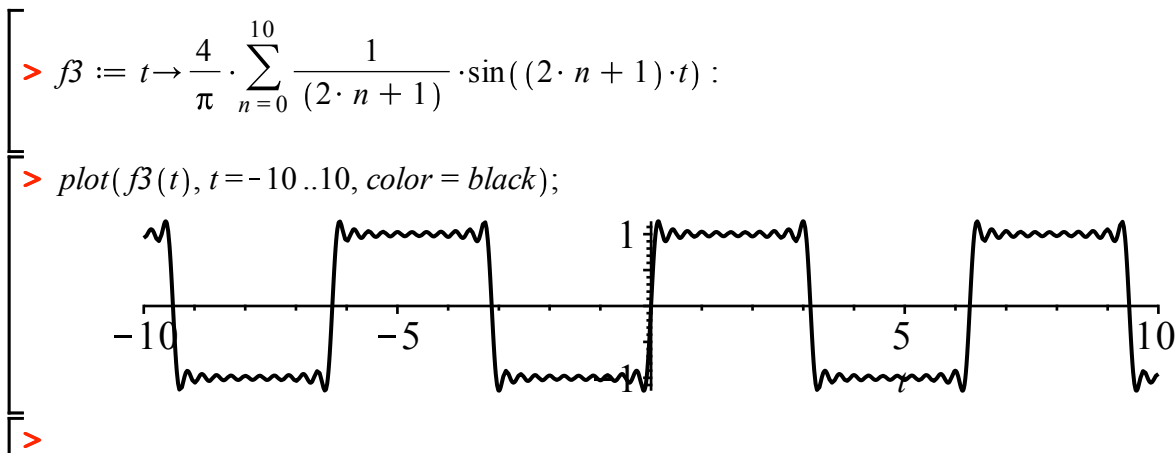
square(t)

(You will find the series directly from the definition in your homework.)

<u>5b</u>) Deduce the magic formula

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ... = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)} = \frac{\pi}{4}.$$

solution: $square \sim \dfrac{4}{\pi} \displaystyle\sum_{n\ odd} \dfrac{1}{n} \sin(nt)$

> $f3 := t \rightarrow \dfrac{4}{\pi} \cdot \displaystyle\sum_{n=0}^{10} \dfrac{1}{(2 \cdot n + 1)} \cdot \sin((2 \cdot n + 1) \cdot t) :$

> $plot(f3(t), t = -10 ..10, color = black);$

>
Could you check the Fourier coefficients with technology?