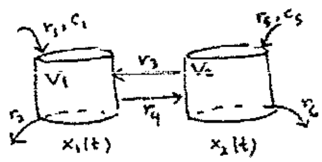
Score	possible
1	(30)
2	(15)
3	(10)
4	(20)
5	(20)
total	(100)

1) Consider a general input-output model with two compartments as indicated below. The compartments contain volumes V_1, V_2 and solute amounts $x_1(t)$, $x_2(t)$ respectively. The flow rates (volume per time) are indicated by r_i, $i = 1..6$. The two input concentrations (solute amount per volume) are c_1, c_5.



1a) Suppose $r_2 = r_3 = 100$, $r_1 = r_4 = r_5 = 200$, $r_6 = 300$ $\frac{m^3}{hour}$. Explain why the volumes $V_1(t)$, $V_2(t)$ remain constant.

(5 points)

$V_1'(t) = r_1 + r_3 - r_2 - r_4 = 200 + 100 - 100 - 200 \equiv 0$

$V_2'(t) = r_5 + r_4 - r_3 - r_6 = 200 + 200 - 100 - 300 \equiv 0$

so $V_1(t)$, $V_2(t)$ are constant.

1b) Using the flow rates above, incoming concentrations $c_1 = 0.05$, $c_5 = 0$ $\frac{kg}{m^3}$, volumes $V_1 = V_2 = 100\ m^3$, show that the amounts of solute $x_1(t)$ in tank 1 and $x_2(t)$ in tank 2 satisfy

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 10 \\ 0 \end{bmatrix}. \quad \text{i.e.} \quad \begin{cases} x_1' = -3x_1 + x_2 + 10 \\ x_2' = 2x_1 - 4x_2 \end{cases}$$

(5 points)

$x_1' = r_1 c_1 + r_3 \frac{x_2}{V_2} - (r_2 + r_4)\frac{x_1}{V_1}$

$= 200(.05) + 100 \frac{x_2}{100} - 300 \frac{x_1}{100}$

$\Rightarrow \boxed{x_1' = 10 + x_2 - 3x_1}$

same

$x_2' = r_5 c_5 + r_4 \frac{x_1}{V_1} - (r_3 + r_6)\frac{x_2}{V_2}$

$= 0 + 200 \frac{x_1}{100} - 400 \frac{x_2}{100}$

$\boxed{x_2' = 2x_1 - 4x_2}$

1c) Find the solution to the homogeneous system of differential equations

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

(10 points)

$$\begin{vmatrix} -3-\lambda & 1 \\ 2 & -4-\lambda \end{vmatrix} = \lambda^2 + 7\lambda + 10 = (\lambda+5)(\lambda+2)$$

$\lambda = -5$: $\left[\begin{array}{cc|c} 2 & 1 & 0 \\ 2 & 1 & 0 \end{array}\right]$

$\vec{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 e^{-5t} \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda = -2$ $\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 2 & -2 & 0 \end{array}\right]$

$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

1d) Find the general solution to the inhomogeneous system of differential equations in 7b,

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 10 \\ 0 \end{bmatrix}.$$

Hint: you will need to find a particular solution as part of your work.

(10 points)

$\vec{x} = \vec{x}_p + \vec{x}_H$

for $\vec{x}_p$ try constant, $\vec{x}_p = \vec{c}$

subs. into DE:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} 10 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} -3 & 1 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -10 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} -4 & -1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} -10 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

so $$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + c_1 e^{-5t} \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

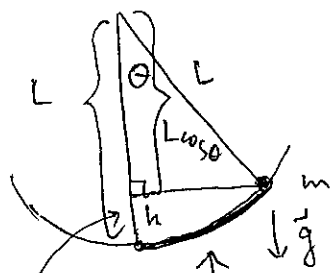
2) We used the principle of conservation of energy and the method of linearization to derive the second order linear differential equation which describes undamped unforced pendulum motion. This differential equation is

$$L\cdot\theta''(t) + g\cdot\theta(t) = 0$$

where $\theta(t)$ is the angle from vertical, L is the length of the (massless) rod or string, m is the mass at the end of the string, and $g = 9.8\ \frac{m}{s^2}$ is the acceleration of gravity on earth.

2a) Derive the linear model above, using the fact that kinetic plus potential energy is constant for any solution to this conservative system. This will yield a non-linear differential equation which you can linearize to the one above, under the assumption that $\theta(t)$ is near zero.

(10 points)



$s = L\theta$
$v = s'(t) = L\theta'(t)$

$h = L - L\cos\theta$

$TE = KE + PE$
$= \frac{1}{2}mv^2 + mgh$
$TE = \frac{1}{2}m\,(L\theta'(t))^2 + mg\,(L - L\cos\theta)$

cons. of energy $TE = \text{const}$
$\Leftrightarrow \frac{d}{dt}TE \equiv 0 = \cancel{\frac{1}{2}}mL^2(\cancel{2}\theta'(t)\theta''(t)) + mg(\cancel{0} - L(-\sin\theta(t))\cdot\theta'(t))$,

$0 \equiv \underbrace{mL\theta'(t)}_{\text{only zero at isolated times}}\left[L\theta''(t) + g\sin\theta(t)\right]$

$\Rightarrow L\theta'' + g\sin\theta = 0$
$\theta'' + \frac{g}{L}\sin\theta = 0$

for θ small $\sin\theta \approx \theta$ yields linearized DE $\boxed{\theta'' + \frac{g}{L}\theta = 0}$

2b) If you wished to create a pendulum for which the natural period was 2 seconds/cycle, what length L would you choose? (A symbolic answer suffices - the correct decimal value for L is close to 1 meter.) Note that you can answer this question from the differential equation given at the start of this problem.

(5 points)

$\omega_0 = \sqrt{\frac{g}{L}}$
$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{L}{g}}$
$\cancel{2} = \cancel{2}\pi\sqrt{\frac{L}{g}}$
$\frac{\sqrt{g}}{\pi} = \sqrt{L}$
$\boxed{L = \frac{g}{\pi^2}}$ ≈ 1 since $g \approx 9.8$, $\pi^2 \approx 9.86$

$$\begin{array}{r} 3.14 \\ \times\ 3.14 \\ \hline 1256 \\ 314 \\ 942 \\ \hline 9.8596 \end{array}$$

3) A focus in this course is a careful analysis of the mathematics and physical phenomena exhibited in forced and unforced mechanical (or electrical) oscillation problems. Using the mass-spring model, we've studied the differential equation for functions $x(t)$ solving

$$m x'' + c x' + k x = F_0 \cos(\omega t)$$

with $m, k, \omega > 0$; $c, F_0 \geq 0$.

Explain what values (or ranges of values) for c, k, F_0, ω lead to the phenomena listed below. (If you use "ω_0" in this discussion make sure to explain what it is as well.) What form will the key parts of the solutions $x(t)$ have in those cases, in order that the physical phenomena be present? (We're not expecting the precise formulas for these parts of the solutions, just what their forms will be.)

3a) pure resonance (3 points)

$F_0 > 0$

$c = 0$

$m x'' + k x = F_0 \cos \omega_0 t$

$\omega = \omega_0 = \sqrt{\frac{k}{m}}$

$x_p = t\,(c_1 \cos \omega_0 t + c_2 \sin \omega_0 t)$ (case II of undetermined coefficients!)

linearly-growing amplitude

3b) beating (3 points)

$F_0 > 0$

$c = 0$

$\omega \approx \omega_0$, $\omega \neq \omega_0$

$m x'' + k x = F_0 \cos \omega t$

$x = C\,(\cos \omega t - \cos \omega_0 t)$

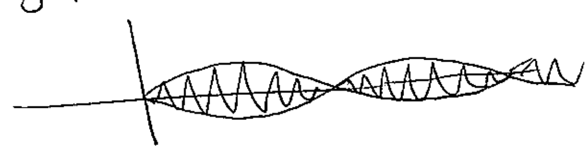
$(+ x_0 \cos \omega_0 t + \frac{v_0}{\omega_0} \sin \omega_0 t)$

@ $t=0$ these cancel out. Because $\omega \neq \omega_0$ after a while they will add up. after twice the time, cancel out. Because

$\cos \omega t - \cos \omega_0 t$

$= D \sin(\frac{\omega - \omega_0}{2} t)(\sin(\frac{\omega + \omega_0}{2} t))$

soltn graphs look like



3c) practical resonance. (4 points)

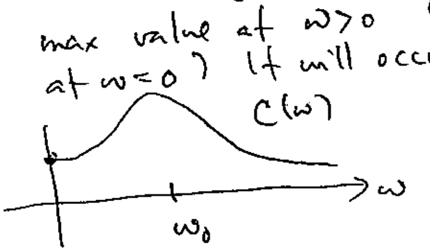
$c > 0$ but $c \approx 0$, $\omega \approx \omega_0$

$x_p = x_{sp} = A \cos \omega t + B \sin \omega t$

$= C \cos(\omega t - \alpha)$

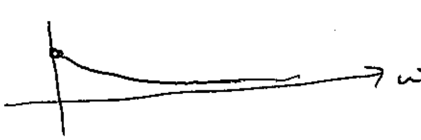
where C is "large".

precisely, text says it's practical resonance if $C = C(\omega)$ has a max value at $\omega > 0$ (as opposed to at $\omega = 0$) It will occur near ω_0

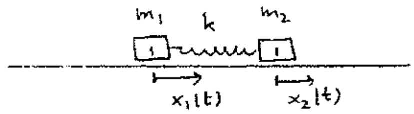


practical resonance.

no practical resonance



4) Consider the following configuration of 2 masses held together with two springs (a 2 car train), with positive displacements from equilibrium for each mass measured to the right, as usual.



4a) Use Newton's Law and Hooke's usual linearization to derive the system of 2 second order differential equations governing the masses' motion.

(5 points)

$$m_1 x_1'' = k(x_2 - x_1)$$
$$m_2 x_2'' = -k(x_2 - x_1)$$

4b) What is the dimension of the solution space to this problem? Explain.

(5 points)

4 - dimensional

- when converted to a 1^{st} order system for x_1, x_1', x_2, x_2' there are 4 homogeneous DE's

(alternately, each natural IVP has 4 initial conditions, $x_1(0), x_1'(0), x_2(0), x_2'(0)$)

4c) Assume that all $m_1 = m_2 = m$. Show that in this case the system in (3a) reduces to

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \frac{k}{m}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(5 points)

if $m_1 = m_2 = m$,

$$4a) \Rightarrow x_1'' = \frac{k}{m}(x_2 - x_1) = \frac{k}{m}(-x_1 + x_2) = -\frac{k}{m}x_1 + \frac{k}{m}x_2$$
$$x_2'' = -\frac{k}{m}(x_2 - x_1) = \frac{k}{m}x_1 - \frac{k}{m}x_2$$

this is the system above

4d) Find the general solution to the differential equation in 3c, repeated here for your convenience. Then describe the general motion as a superposition of two "fundamental modes". Hint: One of the modes is not oscillatory, since this "train" is not tethered to any wall.

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \frac{k}{m}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(10 points)

for $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, $\begin{vmatrix} -1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} = \lambda^2 + 2\lambda = \lambda(\lambda+2)$

evals $\lambda = 0, -2$

$\lambda = 0$: $\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right]$ $\qquad$ $\lambda = -2$ $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right]$

$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. $\qquad$ $\vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

thus for $A = \frac{k}{m}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, $\lambda = 0, -2\frac{k}{m}$

$\downarrow \qquad \downarrow$

$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad \vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\omega = \sqrt{\frac{2k}{m}}$

when $\lambda < 0$, $\omega = \sqrt{-\lambda}$. when $\lambda = 0$ we get solns $(c_1 + c_2 t)\vec{v}$.

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \underbrace{(c_1 + c_2 t)\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \underbrace{\left(c_3 \cos\sqrt{\tfrac{2k}{m}}\,t + c_4 \sin\sqrt{\tfrac{2k}{m}}\,t\right)\begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

spring at equil. length. cars start at $\begin{bmatrix} c_1 \\ c_1 \end{bmatrix}$ and move at constant velocity c_2.

"choo choo"

m_1, m_2 are out of phase, with equal ~~no~~ amplitudes

5) Draw the phase portrait for the first order system, and classify it as one of the following: saddle, nodal source, nodal sink, center, spiral sink, spiral source.

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Hint: Use the eigendata and the form of the general solution.

(20 points)

$$\begin{vmatrix} -4-\lambda & 3 \\ 0 & 6-\lambda \end{vmatrix} = (-4-\lambda)(6-\lambda) = (\lambda+4)(\lambda-6)$$

roots $\lambda = -4, 6$ (know this because matrix is upper Δ'ular, so evals are diagonal entries).

$\lambda = -4$

$$\left[\begin{array}{cc|c} 0 & 3 & 0 \\ 0 & 10 & 0 \end{array}\right]$$

$\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(can see this directly, since $\begin{bmatrix} -4 & 3 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$.

$\lambda = 6$

$$\left[\begin{array}{cc|c} -10 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\vec{v} = \begin{bmatrix} 3 \\ 10 \end{bmatrix}$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 e^{-4t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{6t}\begin{bmatrix} 3 \\ 10 \end{bmatrix}.$$

$\boxed{\text{Saddle}}$

$\lambda_1 < 0 < \lambda_2$