Name..

I.D. number...

Math 2280-001 Spring 2015
PRACTICE FINAL EXAM
(modified from Math 2280 final exam, April 29, 2011)

This exam is closed-book and closed-note. You may use a scientific calculator, but not one which is capable of graphing or of solving differential or linear algebra equations. Laplace Transform and integral tables are included with this exam. **In order to receive full or partial credit on any problem, you must show all of your work** and **justify your conclusions.** This exam counts for 30% of your course grade. It has been written so that there are 150 points possible, and the point values for each problem are indicated in the right-hand margin. **Good Luck!**

problem	score	possible
1	_______	20
2	_______	20
3	_______	30
4	_______	15
5	_______	25
6	_______	15
7	_______	15
8	_______	10
total	_______	150

1) Find the matrix exponentials for the following two matrices. Work one of problems using the power series definition, and the other one using the fundamental matrix solution approach (your choice). As it turns out, both methods are reasonable for both problems.

1a)

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(10 points)

<u>solution via FM:</u> $e^{tA} = \Phi(t)\Phi(0)^{-1}$: For $\Phi(t)$ the columns will form a basis for the solution space to

$$\underline{\boldsymbol{x}}' = A\,\underline{\boldsymbol{x}}$$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1).$$

$E_{\lambda = 1}$:

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right] \Rightarrow \underline{\boldsymbol{v}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ eigenvector}$$

$E_{\lambda = -1}$:

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \Rightarrow \underline{\boldsymbol{v}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ eigenvector}$$

$$\Rightarrow x_H(t) = c_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow \Phi(t) = \begin{bmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{bmatrix} \text{ is an FM}$$

$$\Rightarrow e^{tA} = \Phi(t)\Phi(0)^{-1} = \begin{bmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{bmatrix} \frac{1}{-2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^t + e^{-t} & e^t - e^{-t} \\ e^t - e^{-t} & e^t + e^{-t} \end{bmatrix}.$$

If you try power series you will get entries which are the power series for $\cosh(t)$, $\sinh(t)$. (We did not review those in this 2280 class.) In fact, an equivalent way to write e^{tA} in this case is

$$e^{tA} = \begin{bmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{bmatrix}.$$

1b)

$$B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

(10 points)

<u>with power series:</u>

$$e^{tB} = I + t\,B + \frac{t^2}{2!}B^2 + \ldots + \frac{t^n}{n!}B^n + \ldots$$

Powers of B:

$$B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\ B^2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I,$$

$$B^3 = B^2 B = -I\,B = -B,\ B^4 = B^2 B^2 = (-I)(-I) = I\ B^5 = B^4 B = B\ \ldots$$

and the pattern repeats cyclicly, every four powers. Thus

$$e^{tB}=\begin{bmatrix}1&0\\0&1\end{bmatrix}+t\begin{bmatrix}0&1\\-1&0\end{bmatrix}+\frac{t^2}{2!}\begin{bmatrix}-1&0\\0&-1\end{bmatrix}+\frac{t^3}{3!}\begin{bmatrix}0&-1\\1&0\end{bmatrix}+\frac{t^4}{4!}\begin{bmatrix}1&0\\0&1\end{bmatrix}+\ldots$$

$$=\begin{bmatrix}1-\frac{t^2}{2!}+\frac{t^4}{4!}+\ldots & t-\frac{t^3}{3!}+\frac{t^5}{5!}-\ldots\\ -t+\frac{t^3}{3!}-\frac{t^5}{5!}+\ldots & 1-\frac{t^2}{2!}+\frac{t^4}{4!}+\ldots\end{bmatrix}=\begin{bmatrix}\cos(t)&\sin(t)\\-\sin(t)&\cos(t)\end{bmatrix}.$$

<u>solution via FM:</u> $e^{tA}=\Phi(t)\Phi(0)^{-1}$: For $\Phi(t)$the columns will form a basis for the solution space to

$$\underline{\boldsymbol{x}}'=A\,\underline{\boldsymbol{x}}$$

$$|A-\lambda I|=\begin{vmatrix}-\lambda&1\\-1&-\lambda\end{vmatrix}=\lambda^2+1=(\lambda-i)(\lambda+i).$$

$E_{\lambda=i}$:

$$\left[\begin{array}{cc|c}-i&1&0\\-1&-i&0\end{array}\right]\Rightarrow\underline{\boldsymbol{v}}=\begin{bmatrix}1\\i\end{bmatrix}\text{ eigenvector}$$

complex solution

$$\underline{z}(t)=e^{it}\begin{bmatrix}1\\i\end{bmatrix}=(\cos(t)+i\sin(t))\begin{bmatrix}1\\i\end{bmatrix}=\begin{bmatrix}\cos(t)\\-\sin(t)\end{bmatrix}+i\begin{bmatrix}\sin(t)\\\cos(t)\end{bmatrix}.$$

The real and complex parts are each real solutions, so a FM is given by

$$\Phi(t)=\begin{bmatrix}\cos(t)&\sin(t)\\-\sin(t)&\cos(t)\end{bmatrix}.$$

Since $\Phi(0)=I$, this is e^{tA}

$$e^{tA}=\begin{bmatrix}\cos(t)&\sin(t)\\-\sin(t)&\cos(t)\end{bmatrix}.$$

2a) Use Laplace transform techniques to find the general solution to the undamped forced oscillator equation with resonance:

$$x''(t)+\omega_0^2\,x(t)=F_0\,\sin(\omega_0\,t).$$

(10 points)

<u>solution:</u> The solution $x(t)$ makes both sides of the DE equal, so their Laplace transforms are too.

$$s^2X(s)-s\,x_0-v_0+\omega_0^2X(s)=F_0\frac{\omega_0}{s^2+\omega_0^2}$$

$$X(s)\left(s^2+\omega_0^2\right)=F_0\frac{\omega_0}{s^2+\omega_0^2}+s\,x_0+v_0$$

$$X(s)=F_0\frac{\omega_0}{\left(s^2+\omega_0^2\right)^2}+x_0\frac{s}{s^2+\omega_0^2}+v_0\frac{1}{s^2+\omega_0^2}$$

$$x(t)=F_0\omega_0\left(\frac{1}{2\,\omega_0^3}\right)\left(\sin(\omega_0t)-\omega_0t\cos(\omega_0t)\right)+x_0\cos(\omega_0t)+\frac{v_0}{\omega_0}\sin(\omega_0t)$$

$$=F_0\left(\frac{1}{2\,\omega_0^2}\right)\left(\sin(\omega_0t)-\omega_0t\cos(\omega_0t)\right)+x_0\cos(\omega_0t)+\frac{v_0}{\omega_0}\sin(\omega_0t).$$

2b) Use Laplace transform to find the general solution to the non-resonant undamped forced oscillator equation

$$x''(t) + \omega_0^2 x(t) = F_0 \sin(\omega t)$$
$$\omega \neq \omega_0$$

(10 points)

solution:

$$s^2 X(s) - s x_0 - v_0 + \omega_0^2 X(s) = F_0 \frac{\omega}{s^2 + \omega^2}$$

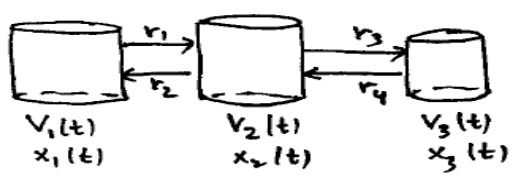
$$X(s)\left(s^2 + \omega_0^2\right) = F_0 \frac{\omega}{s^2 + \omega^2} + s x_0 + v_0$$

$$X(s) = F_0 \omega \frac{1}{\left(s^2 + \omega_0^2\right)\left(s^2 + \omega^2\right)} + x_0 \frac{s}{s^2 + \omega_0^2} + v_0 \frac{1}{s^2 + \omega_0^2}$$

$$X(s) = F_0 \omega \frac{1}{\omega^2 - \omega_0^2}\left(\frac{1}{s^2 + \omega_0^2} - \frac{1}{s^2 + \omega^2}\right) + x_0 \frac{s}{s^2 + \omega_0^2} + v_0 \frac{1}{s^2 + \omega_0^2}$$

$$x(t) = F_0 \omega \frac{1}{\omega^2 - \omega_0^2}\left(\frac{1}{\omega_0}\sin(\omega_0 t) - \frac{1}{\omega}\sin(\omega t)\right) + x_0 \cos(\omega_0 t) + \frac{v_0}{\omega_0}\sin(\omega_0 t).$$

3) Consider the following three-tank configuration. Let tank i have volume $V_i(t)$ and solute amount $x_i(t)$ at time t. Well-mixed liquid flows between tanks one and two, with rates r_1, r_2, and also between tanks two and three, with rates r_3, r_4, as indicated.



3a) What is the system of 6 first order differential equations governing the volumes $V_1(t)$, $V_2(t)$, $V_3(t)$ and solute amounts $x_1(t)$, $x_2(t)$, $x_3(t)$? (Hint: Although most of our recent tanks have had constant volume, we've also discussed how to figure out how fast volume is changing in input/output models.

(6 points)

$$V_1'(t) = r_2 - r_1$$
$$V_2'(t) = r_1 + r_4 - r_2 - r_3$$
$$V_3'(t) = r_3 - r_4$$
$$x_1'(t) = -r_1\left(\frac{x_1}{V_1}\right) + r_2\left(\frac{x_2}{V_2}\right)$$
$$x_2'(t) = r_1\left(\frac{x_1}{V_1}\right) - (r_2 + r_3)\left(\frac{x_2}{V_2}\right) + r_4\left(\frac{x_3}{V_3}\right)$$
$$x_3'(t) = r_3\left(\frac{x_2}{V_2}\right) - r_4\left(\frac{x_3}{V_3}\right).$$

3b) Suppose that all four rates are 100 gallons/hour, so that the volumes in each tank remain constant. Suppose that these volumes are each 100 gallons. Show that in this case, the differential equations in (2a) *for the solute amounts reduce to the system*

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(4 points)

<u>solution:</u> In this case, each $\frac{r_i}{V_i}$ has numerical value $\frac{100}{100} = 1$ so the differential equations for the x_j simplify to

$$x_1'(t) = -r_1\left(\frac{x_1}{V_1}\right) + r_2\left(\frac{x_2}{V_2}\right) = -x_1 + x_2$$

$$x_2'(t) = r_1\left(\frac{x_1}{V_1}\right) - (r_2 + r_3)\left(\frac{x_2}{V_2}\right) + r_4\left(\frac{x_3}{V_3}\right) = x_1 - 2\,x_2 + x_3$$

$$x_3'(t) = r_3\left(\frac{x_2}{V_2}\right) - r_4\left(\frac{x_3}{V_3}\right) = x_2 - x_3$$

which can be rewritten in the matrix vector form displayed above.

3c) Maple to the rescue! Maple says that

```
> with(LinearAlgebra) :
> A := Matrix(3, 3, [-1, 1, 0, 1,-2, 1, 0, 1,-1]);
  Eigenvectors(A);
```

$$A := \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix} \qquad \textbf{(1)}$$

Use this information to write the general solution to the system in (3b).

(5 points)

<u>solution</u>: The eigenvalues are in the first column of output, and the corresponding eigenvectors are in the columns of the matrix. Each eigenpair $(\lambda, \underline{\boldsymbol{v}})$ yields a solution $e^{\lambda t}\underline{\boldsymbol{v}}$ so the general solution to $\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$ is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{-3t} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

3d) Solve the initial value problem for the tank problem in (3b), assuming there are initially 10 pounds of solute in tank 1, 20 pounds in tank 2, and none in tank 3.

(10 points)

Using the solution above, at $t = 0$:

$$\begin{bmatrix} 10 \\ 20 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 10 \\ 1 & 0 & -2 & 20 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

$-R_1 + R_2 \to R_2,\ -R_1 + R_3 \to R_3$:

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 10 \\ 0 & 1 & -3 & 10 \\ 0 & 2 & 0 & -10 \end{array}\right].$$

$-2\,R_2 + R_3 \to R_3$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 10 \\ 0 & 1 & -3 & 10 \\ 0 & 0 & 6 & -30 \end{array}\right].$$

$\frac{R_3}{6} \to R_3$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 10 \\ 0 & 1 & -3 & 10 \\ 0 & 0 & 1 & -5 \end{array}\right].$$

$3\,R_3 + R_2 \to R_2,\ -R_3 + R_1 \to R_1$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 15 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -5 \end{array}\right].$$

$R_2 + R_1 \to R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -5 \end{array}\right].$$

so $c_1 = 10,\ c_2 = -5,\ c_3 = -5$ and

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = 10 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + -5e^{-t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + -5e^{-3t} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

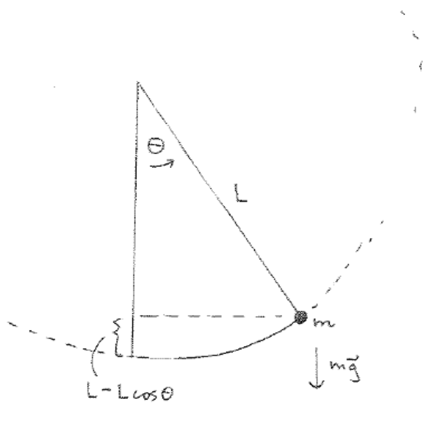
3e) What is the limiting amount of salt in each tank, as t approaches infinity? (Hint: You can deduce this answer, no matter whether you actually solved 4d, but this gives a way of partially checking your work there.)

(5 points)

Since there are 30 pounds total of salt and since the tanks each have the same volume, as the concentrations converge to their final uniform concentration, the salt amounts will converge to 10 pounds per tank. This is also clear from the solution formula, as the second two terms decay exponentially to zero.

4) Although we usually use a mass-spring configuration to give context for studying second order differential equations, the rigid-rod pendulum also effectively exhibits several key ideas from this course.

Recall that in the undamped version of this configuration, we let the pendulum rod length be L, assume the rod is massless, and that there is a mass m attached at the end on which the vertical graviational force acts with force $m \cdot g$. This mass will swing in a circular arc of signed arclength $s = L \cdot \theta$ from the vertical, where θ is the angle in radians from vertical. The configuration is indicated below.



4a) Use the fact that the undamped system is conservative, to derive the differential equation for $\theta(t)$,

$$\theta''(t) + \frac{g}{L} \cdot sin(\theta(t)) = 0.$$

(10 points)

Hint: Begin by express the TE=KE+PE in terms of the function $\theta(t)$ and its derivatives. Then compute $TE'(t)$ and set it equal to zero.

$$TE = KE + PE = \frac{1}{2} m v^2 + mgh.$$

Measure the arclength s from the bottom to the mass location, and it's given by $s = L\,\theta$. The scalar velocity is $v(t) = s'(t) = L\theta'(t)$. Measure height from the bottom and it is given by $h = L - L\cos(\theta)$. Thus

$$TE(t) = \frac{1}{2} m (L\theta'(t))^2 + m\,g\,(L - L\cos(\theta(t))).$$

Total energy constant is equivalent to $TE'(t) \equiv 0$, i.e.

$$0 \equiv \frac{1}{2} mL^2 2\,\theta'(t)\theta''(t) + mgL\sin(\theta(t))\theta'(t)$$

$$0 \equiv mL\theta'(t)(L\,\theta''(t) + g\sin(\theta(t)).$$

Since $\theta'(t)$ can only be zero at isolated times, it must be that

$$0 \equiv L\,\theta''(t) + g\sin(\theta(t)$$

which is the same as the claimed DE, if we divide both sides by L.

4b) Explain precisely how the second order differential equation in (5a) is related to the first order system of differential equations

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} y \\ -\frac{g\,sin(x)}{L} \end{bmatrix}$$

(5 points)

solution: Let $\theta(t)$ solve

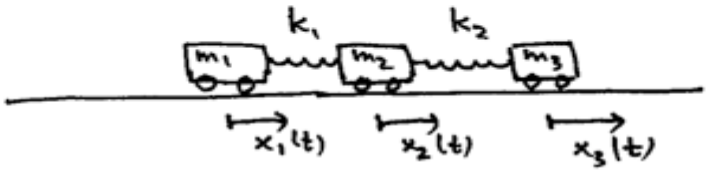
$$\theta''(t) + \frac{g}{L} \cdot sin(\theta(t)) = 0.$$

Define $x(t) := \theta(t), y(t) = \theta'(t)$. Then

$$x'(t) = \theta'(t) = y$$
$$y'(t) = \theta''(t) = -\frac{g}{L}\sin(\theta(t)) = -\frac{g}{L}\sin(x(t))$$

which is the displayed system. (Conversely, if $[x(t), y(t)]^T$ solve the system, then defining $\theta(t) := x(t)$ yields a solution to the second order DE for $\theta(t)$.)

5) Consider the following 3-mass, 2-spring zero-drag "train"configuration below. At rest the cars are separated by certain distances and the springs are neither pulling nor pushing. From that equilibrium configuration, the train is put pushed into motion along a track, and the displacements from equilbrium are measured by $x_1(t)$, $x_2(t)$, $x_3(t)$ as indicated.



5a) Use Newton's law and the Hooke's law (linearization), to derive the system of differential equations for $x_1(t)$, $x_2(t)$, $x_3(t)$.

(8 points)

$$m_1 x_1''(t) = k_1(x_2 - x_1)$$
$$m_2 x_2''(t) = -k_1(x_2 - x_1) + k_2(x_3 - x_2)$$
$$m_3 x_3''(t) = -k_2(x_3 - x_2).$$

5b) Show that in case units are chosen so that the numerical values of all the masses are the same as the numerical values of the spring, i.e. $m_1 = m_2 = m_3 = k_1 = k_2 = k_3$ then the system above reduces to

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \\ x_3''(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(4 points)

In this case we may divide each of the DE's in 5a by the corresponding mass, and replace each $\frac{k_j}{m_j}$ by 1.

This yields

$$x_1''(t) = (x_2 - x_1) = -x_1 + x_2$$
$$x_2''(t) = -(x_2 - x_1) + (x_3 - x_2) = x_1 - 2x_2 + x_3$$
$$x_3''(t) = -(x_3 - x_2) = x_2 - x_3$$

which is equivalent to the matrix vector system that is displayed.

5c) Exhibit the general solution for the system in 5b. Note that you've already seen this matrix in problem 3:

```
> with(LinearAlgebra) :
> A := Matrix(3, 3, [-1, 1, 0, 1,-2, 1, 0, 1,-1]);
  Eigenvectors(A);
```

$$A := \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix} \qquad \textbf{(2)}$$

(8 points)

solution: Recall (and you should be able to explain why) that if $(\lambda, \underline{\boldsymbol{v}})$ is an eigenpair for the matrix A, with $\lambda < 0$, then for $\omega = \sqrt{-\lambda}$ we get solutions $\cos(\omega\, t)\underline{\boldsymbol{v}}$, $\sin(\omega\, t)\underline{\boldsymbol{v}}$. If $\lambda = 0$ we get solutions $\underline{\boldsymbol{v}}$, $t\,\underline{\boldsymbol{v}}$. So,

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = (c_1 + c_2\, t)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (c_3 \cos(t) + c_4 \sin(t))\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \left(c_5 \cos\left(\sqrt{3}\, t\right) + c_6 \sin\left(\sqrt{3}\, t\right)\right)\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

5d) Describe the general motion of the train as a superposition of three fundamental modes.

(5 points)

In the first mode,

$$(c_1 + c_2\, t)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The train is moving without oscillation, having started at c_1 from the chosen origin, and with velocity c_2.

In the second mode,

$$(c_3 \cos(t) + c_4 \sin(t))\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

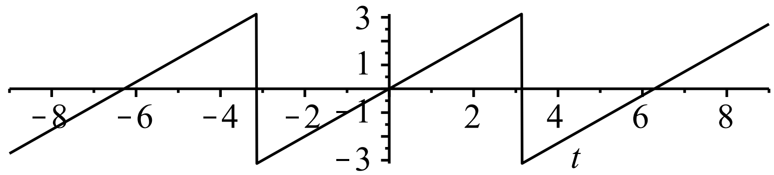
the first and third cars are oscillating out of phase and with equal amplitudes, while the second car remains stationary.

In the third mode

$$\left(c_5 \cos\left(\sqrt{3}\, t\right) + c_6 \sin\left(\sqrt{3}\, t\right)\right)\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

the cars are oscillating the most rapidly, with the first and third cars in phase with equal amplitude, and the middle car out of phase, with twice the amplitude of the outer two cars.

6a) We consider a 2 π-periodic saw-tooth function, given on the interval $(-\pi, \pi)$ by $f(t) = t$, and equal to zero at every integer multiple of π.
Here's a graph of a piece of this function:



Derive the Fourier series for $f(t)$,

$$f(t) = 2\left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sin(n\,t)}{n}\right).$$

(10 points)

<u>solution:</u> Because $f(t)$ is an odd function, its Fourier cosine coefficients are all zero.

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n\frac{\pi}{L}t\right) + \sum_{n=1}^{\infty} b_n \sin\left(n\frac{\pi}{L}t\right) = \sum_{n=1}^{\infty} b_n \sin\left(n\frac{\pi}{L}t\right)$$

with

$$b_n := \left\langle f, \sin\left(n\frac{\pi}{L}\,t\right)\right\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(n\,t)\,\mathrm{d}t = \frac{2}{\pi}\int_0^{\pi} t\sin(n\,t)\,\mathrm{d}t.$$

(The last step holds because the integrand is odd*odd=even function.) Integrate by parts, letting $u = t$, $du = dt$, $dv = \sin(n\,t)$, $v = -\frac{1}{n}\cos(n\,t)$

$$\frac{2}{\pi}\int_0^{\pi} t\sin(n\,t)\,\mathrm{d}t = \frac{2}{\pi}\left(\left[-\frac{t}{n}\cos(n\,t)\right]_0^{\pi} = \int_0^{\pi} -\frac{1}{n}\cos(n\,t)\,\mathrm{d}t\right)$$

$$= \frac{2}{\pi}\left(-\frac{\pi}{n}\cos(n\pi) + \left[\frac{1}{n^2}\sin(n\,t)\right]_0^{\pi}\right)$$

$$= -\frac{2}{n}\left((-1)^n - 0\right) = \frac{2}{n}(-1)^{n+1}.$$

Setting $b_n = \frac{2}{n}(-1)^{n+1}$, $a_0 = 0$, $a_n = 0$ yields the displayed Fourier series

$$f(t) = 2\left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sin(n\,t)}{n}\right).$$

6b) Use the Fourier series above to explain the identity

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots.$$

(5 points)

<u>solution</u> Since $f(t)$ is differentiable at $t = \frac{\pi}{2}$ the Fourier series converges to $f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$ there, i.e.

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} = 2\left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin\left(\frac{n\pi}{2}\right)}{n}\right).$$

Since $\sin\left(\frac{n\pi}{2}\right) = 0$ for n even, $\sin\left(\frac{n\pi}{2}\right) = +1$ for $n = 1 + 4k$, $k \in \mathbb{N}$, $\sin\left(\frac{n\pi}{2}\right) = -1$ for $n = 3 + 4k$, $k \in \mathbb{N}$, and since $(-1)^{n+1} = 1$ for n odd, we get

$$\frac{\pi}{2} = 2\left(1 - \frac{1}{3} + \frac{1}{5} - \ldots\right).$$

Divide both sides by 2 to get the displayed identity.

7) Consider the saw-tooth function $f(t)$ from problem 6, and the forced oscillation problem

$$x''(t) + 9 \cdot x(t) = f(t).$$

7a) Discuss whether or not resonance occurs.

(5 points)

The natural angular frequency is $\omega_0 = 3$. Since the Fourier expansion of $f(t)$ has a $\sin(3t)$ term, there will be resonance.

7b) Find a particular solution for this forced oscillation problem. Hint: Use the Fourier series for $f(t)$ given in problem 6. You may make use of the particular solutions table on the next page

(10 points)

We use (infinite) superposition to find a particular solution.

$$x''(t) + 9 \cdot x(t) = 2\left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin(n t)}{n}\right).$$

For $n \neq 3$ the forced oscillation problem

$$x''(t) + 9 \cdot x(t) = \frac{2}{n}(-1)^{n+1} \sin(n t)$$

has particular solution

$$x_P(t) = \frac{2}{n}(-1)^{n+1}\left(\frac{1}{9 - n^2}\sin(nt)\right).$$

For $n = 3$ the forced oscillation problem

$$x''(t) + 9 \cdot x(t) = \frac{2}{3}\sin(3t)$$

has a particular solution

$$x_P(t) = \frac{2}{3}\left(-\frac{t}{6}\cos(3t)\right) = -\frac{1}{9} t\cos(3t).$$

Thus for

$$x''(t) + 9 \cdot x(t) = 2\left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin(n t)}{n}\right)$$

we have a particular solution

$$x_P(t) = -\frac{1}{9} t\cos(3t) + 2\left(\sum_{n \neq 3} \frac{(-1)^{n+1} \sin(n t)}{n\left(9 - n^2\right)}\right).$$

(The sum on the right converges to a bounded function, with absolute value less that

$$\sum_{n \neq 3} \frac{1}{n|9 - n^2|} < \infty$$

by ratio comparison to the convergent series

$$\sum_{n=1}^{\infty} \frac{1}{n^3}.)$$

8) We discussed the analogy between constant coefficient first-order linear differential equations (in Chapter 1), and first order systems of differential equations (In Chapter 5). Use matrix exponentials and the "integrating factor" technique to show that for first order systems with constant matrix A, the general solution to

$$\underline{\mathbf{x}}'(t) = A\,\underline{\mathbf{x}} + \underline{\mathbf{f}}(t)$$

is given by the formula

$$\underline{\mathbf{x}}(t) = e^{tA}\left(\int e^{-tA}\underline{\mathbf{f}}(t)\,dt\right) + e^{tA}\underline{\mathbf{c}}.$$

(In the formula above, $\int e^{-tA}\underline{\mathbf{f}}(t)\,dt$ *is standing for any particular antiderivative of* $e^{-tA}\underline{\mathbf{f}}(t)$, *and the displayed formula is expressing* $\underline{\mathbf{x}}(t)$ *as* $\underline{\mathbf{x}}_P + \underline{\mathbf{x}}_H$*)*

<u>Hint</u>: begin by rewriting the system as

$$\underline{\mathbf{x}}'(t) - A\,\underline{\mathbf{x}} = \underline{\mathbf{f}}(t)$$

and then find an appropriate (matrix) integrating factor.

(10 points)

<u>solution:</u>

$$\underline{\mathbf{x}}'(t) - A\,\underline{\mathbf{x}} = \underline{\mathbf{f}}(t)$$
$$\Rightarrow \mathrm{e}^{-tA}(\underline{\mathbf{x}}'(t) - A\,\underline{x}) = \mathrm{e}^{-tA}\underline{\mathbf{f}}(t)$$
$$\Rightarrow \frac{d}{dt}\left(\mathrm{e}^{-tA}\underline{\mathbf{x}}(t)\right) = \mathrm{e}^{-tA}\underline{\mathbf{f}}(t)$$
$$\Rightarrow \mathrm{e}^{-tA}\underline{\mathbf{f}}(t) = \int e^{-tA}\underline{\mathbf{f}}(t)\,dt + \underline{\mathbf{c}}$$
$$\Rightarrow \underline{\mathbf{x}}(t) = e^{tA}\left(\int e^{-tA}\underline{\mathbf{f}}(t)\,dt\right) + e^{tA}\underline{\mathbf{c}} = \underline{\mathbf{x}}_P + \underline{\mathbf{x}}_H$$

Note: One uses the "universal" product rule we discussed in class, and

$$\frac{d}{dt}\mathrm{e}^{-tA} = \mathrm{e}^{-tA}A$$

to justify

$$\frac{d}{dt}\left(\mathrm{e}^{-tA}\underline{\mathbf{x}}(t)\right) = \mathrm{e}^{-tA}(\underline{\mathbf{x}}'(t) - A\,\underline{x}).$$

Fourier series information: For $f(t)$ of period $P = 2L$,

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n \frac{\pi}{L} t\right) + \sum_{n=1}^{\infty} b_n \sin\left(n \frac{\pi}{L} t\right)$$

with

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\,dt \qquad \left(\text{so } \frac{a_0}{2} = \frac{1}{2L}\int_{-L}^{L} f(t)\,dt \text{ is the average value of } f\right)$$

$$a_n := \left\langle f, \cos\left(n \frac{\pi}{L} t\right)\right\rangle = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(n \frac{\pi}{L} t\right) dt, \quad n \in \mathbb{N}$$

$$b_n := \left\langle f, \sin\left(n \frac{\pi}{L} t\right)\right\rangle = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(n \frac{\pi}{L} t\right) dt, \quad n \in \mathbb{N}$$

Particular solutions from Chapter 3 or Laplace transform table:

$$x''(t) + \omega_0^2 x(t) = A\, sin(\omega t)$$

$$x_P(t) = \frac{A}{\omega_0^2 - \omega^2}\sin(\omega t) \qquad \text{when } \omega \neq \omega_0$$

$$x_P(t) = -\frac{t}{2\omega_0} A \cos(\omega_0 t) \qquad \text{when } \omega = \omega_0$$

....

$$x''(t) + \omega_0^2 x(t) = A \cos(\omega t)$$

$$x_P(t) = \frac{A}{\omega_0^2 - \omega^2}\cos(\omega t) \qquad \text{when } \omega \neq \omega_0$$

$$x_P(t) = \frac{t}{2\omega_0} A \sin(\omega_0 t) \qquad \text{when } \omega = \omega_0$$

....

$$x'' + c\,x' + \omega_0^2 x = A \cos(\omega t) \qquad c > 0$$

$$x_P(t) = x_{sp}(t) = C \cos(\omega t - \alpha)$$

with

$$C = \frac{A}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + c^2\omega^2}} .$$

$$\cos(\alpha) = \frac{\omega_0^2 - \omega^2}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + c^2\omega^2}}$$

$$\sin(\alpha) = \frac{c\,\omega}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + c^2\omega^2}} .$$

....

$$x'' + c\,x' + \omega_0^2 x = A \sin(\omega t) \qquad c > 0$$

$$x_P(t) = x_{sp}(t) = C \sin(\omega t - \alpha)$$

with

$$C = \frac{A}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + c^2\omega^2}} .$$

$$\cos(\alpha) = \frac{\omega^2 - \omega_0^2}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + c^2\omega^2}}$$

$$\sin(\alpha) = \frac{c\,\omega}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + c^2\omega^2}}$$

Table of Laplace Transforms

This table summarizes the general properties of Laplace transforms and the Laplace transforms of particular functions derived in Chapter 10.

Function	Transform	Function	Transform
$f(t)$	$F(s)$	e^{at}	$\frac{1}{s-a}$
$af(t)+bg(t)$	$aF(s)+bG(s)$	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$f'(t)$	$sF(s)-f(0)$	$\cos kt$	$\frac{s}{s^2+k^2}$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$	$\sin kt$	$\frac{k}{s^2+k^2}$
$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$	$\cosh kt$	$\frac{s}{s^2-k^2}$
$\int_0^t f(\tau)\,d\tau$	$\frac{F(s)}{s}$	$\sinh kt$	$\frac{k}{s^2-k^2}$
$e^{at}f(t)$	$F(s-a)$	$e^{at}\cos kt$	$\frac{s-a}{(s-a)^2+k^2}$
$u(t-a)f(t-a)$	$e^{-as}F(s)$	$e^{at}\sin kt$	$\frac{k}{(s-a)^2+k^2}$
$\int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$	$\frac{1}{2k^3}(\sin kt - kt\cos kt)$	$\frac{1}{(s^2+k^2)^2}$
$tf(t)$	$-F'(s)$	$\frac{t}{2k}\sin kt$	$\frac{s}{(s^2+k^2)^2}$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	$\frac{1}{2k}(\sin kt + kt\cos kt)$	$\frac{s^2}{(s^2+k^2)^2}$
$\frac{f(t)}{t}$	$\int_s^\infty F(\sigma)\,d\sigma$	$u(t-a)$	$\frac{e^{-as}}{s}$
$f(t)$, period p	$\frac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)\,dt$	$\delta(t-a)$	e^{-as}
1	$\frac{1}{s}$	$(-1)^{[\![t/a]\!]}$ (square wave)	$\frac{1}{s}\tanh\frac{as}{2}$
t	$\frac{1}{s^2}$	$\left[\!\left[\frac{t}{a}\right]\!\right]$ (staircase)	$\frac{e^{-as}}{s(1-e^{-as})}$
t^n	$\frac{n!}{s^{n+1}}$		
$\frac{1}{\sqrt{\pi t}}$	$\frac{1}{\sqrt{s}}$		
t^a	$\frac{\Gamma(a+1)}{s^{a+1}}$		