

Recall that all listed problems are good for seeing if you can work with the underlying concepts; that the underlined problems are to be handed in; and that the Friday quiz will be drawn from all of these concepts and from these or related problems.

3.4: *applications for unforced mechanical systems.*

3.4: 5, 6, 23

week 7.1 Energy in a mass-spring-damper system: Let $x(t)$ be the position of a mass m attached to a spring with Hooke's constant k and damping piston with constant c , yielding the differential equation

$$m x'' + c x' + kx = f(t),$$

where $f(t)$ is an external forcing on the mass. We wish to account for the total energy of the mass-spring configuration, neglecting the heat energy loss due to damping. We define the total energy to be the sum of kinetic and potential energy. Potential energy $PE(t)$ is stored by the compressed or stretched spring, and is the work done to stretch/compress it as the mass moves from the equilibrium $x = 0$, to position x :

$$PE(t) = \int_0^x ku \, du = \frac{k}{2}x^2.$$

As usual, the kinetic energy of the mass is

$$KE(t) = \frac{m}{2}(x')^2.$$

The sum is the total energy

$$E(t) = PE(t) + KE(t) = \frac{1}{2}(kx(t)^2 + m(x'(t))^2).$$

- a)** Take the derivative of $E(t)$ with respect to time. Use the chain rule on the right side of the equation. Use the original differential equation to substitute for $x''(t)$, and simplify your work so that you get a formula for $E'(t)$ that only depends on the forcing $f(t)$, the velocity $x'(t)$, and the damping coefficient c .
- b)** Assume there is no external forcing, i.e. $f = 0$. In this case, what condition(s) on m , c , k guarantee that the energy in the system is constant (i.e. $E'(t) = 0$).
- c)** Set $f = 0$, $m = 1$, $c = 2$, $k = 5$. Set initial conditions $x(0) = 2$ and $x'(0) = 0$. How long will it take for the system to lose 80 % of its initial energy? To solve, find the solution $x(t)$ to the IVP and use this solution to compute the energy function E explicitly.
- d)** Plot the energy curve (with technology) and describe its behavior. Explain in words the mechanism that drives the energy picture that you observe, and why the energy is not decreasing at the initial instant $t = 0$, but then decreases more rapidly later on.

3.5: *finding particular solutions using the method of undetermined coefficients; using variation of parameters; using the general solution $y = y_p + y_H$ to solve associated initial value problems.*

3.5: 2, 3, 10, 21, 27, 29, 31, 34.

week 7.2) Consider the 3^{rd} order differential operator for $y(x)$:

$$L(y) := y''' - 3y' + 2y.$$

a) Find the solution space to the homogeneous differential equation $L(y) = 0$. Hint: first find an integer root of the characteristic polynomial, then do long division.

b) Use the method of undetermined coefficients to find a particular solution to $L(y) = x$.

c) Use the method of undetermined coefficients to find a particular solution to $L(y) = e^{3x}$.

d) Use the method of undetermined coefficients to find a particular solution to $L(y) = e^{-2x}$. Hint: this will involve a lot of product rule differentiation if you just plug in the correct undetermined coefficients trial solution. An alternate shortcut you might consider would be to factor L so that $(D + 2I)$ is one of the factors and is the part of L you apply first. We do an example like this on Monday February 23.

e) Use your work in a,b,c,d and linearity (superposition) to write down the general solution to

$$L(y) = 8x + 20e^{3x} - 8e^{-2x}.$$

f) Hand in a technology check (e.g. Maple or Wolfram alpha) for your answer to **e**.

3.5: *non-standard right-hand sides; variation of parameters.*

3.5: **43**, 45, 47, **52**, 57, **58**.

week 7.3) Reconsider problem **52** above, from the point of view of undetermined coefficients, but taking advantage of complex exponentials and factorization of operators. The problem is to find a particular solution to

$$L(y) := y'' + 9y = \sin(3x).$$

Since the roots of the characteristic polynomial $p(r) = r^2 + 9$ are $r = \pm 3i$, the term on the right, $\sin(3x)$ solves the homogeneous DE, and Case II of undetermined coefficients prescribes an undetermined coefficients test solution of the form

$$y_p(x) = x(A \cos(3x) + B \sin(3x)).$$

Carrying this procedure out works, but is somewhat messy. Rather, factor L as

$$L = D^2 + 9I = (D + 3iI) \circ (D - 3iI).$$

Recall from previous homework that $(D - rI)f(x)e^{rx} = f'(x)e^{rx}$.

a) Use the factorization above, and the fact above, to compute that

$$L(xe^{3ix}) = 6ie^{3ix}.$$

b) Use Euler's formula and linearity to deduce from the real and imaginary parts of the formula in **a** that

$$L(x \cos(3x)) = -6 \sin(3x)$$

$$L(x \sin(3x)) = 6 \cos(3x).$$

c) Use linearity to deduce that a particular solution to the original differential equation is

$$y_p(x) = -\frac{1}{6}x \cos(3x).$$

week 7.4) (This does not need to be handed in.) Read and understand the first proof of the rank+nullity theorem in Wikipedia. As we discuss in class on Monday, we are really using a special case of this theorem in justifying the method of undetermined coefficients.

http://en.wikipedia.org/wiki/Rank-nullity_theorem