Math 2280-001
Week 6 concepts and homework, due Friday February 20, at the start of class
(Our first midterm is also on Friday)

Recall that all problems are good for seeing if you can work with the underlying concepts; that the underlined problems are to be handed in; and that the Friday quiz will be drawn from all of these concepts and from these or related problems.

3.3: using the algorithm for finding the general solution to constant coefficient linear homogeneous differential equations: real roots, Euler's formula and complex roots, repeated roots; solving associated initial value problems.

week 5.1) We explore why the general algorithm for finding a basis of solutions for constant coefficient homogeneous linear differential equations yields independent functions, for an illustrative example.

a) What 5th order constant coefficient homogeneous linear DE has characteristic polynomial
$$p(r) = (r-3)^3 (r^2 + 16) \ ?$$

b) According to our general algorithm, write down a basis for the 5-dimensional solution space to the DE in part **a**.

c) Prove that the five functions you wrote down in part **b** actually are linearly independent (so automatically span, so a basis). Hint: Use a combination of the techniques we discussed on Wednesday February 11, rather than attempting a Wronskian computation. Notice that the solutions exist on the entire real line $-\infty < x < \infty$ and some of them have different growth rates.

In the next exercise you will continue to explore the reasoning that leads to the "magic" general algorithm for solving constant coefficient homogeneous linear DEs in the case that the characteristic polynomial has repeated roots. These ideas are also discussed in section 3.3, and it may help you to read the book's discussion. Here's a summary: Begin with the derivative operator D defined by $D(y) := y'$. Then the second derivative operator is $D^2(y) := D \circ D(y)$, etc. So for $y = y(x)$ and
$$L(y) := y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_1 y' + a_0 y$$
we may write
$$L = D^n + a_{n-1} D^{n-1} + ... + a_1 D + a_0 I$$
where I is the identity operator, $I(y) := y$. In other words, we are writing the operator L as a linear combination of various derivative operators. It turns out that any factorization of the characteristic polynomial $p(r)$ leads to a corresponding factorization of L. For example, consider the second order case with real roots,
$$L = D^2 + a_1 D + a_0 I.$$
If $p(r)$ factors with real roots,

$$p(r) = r^2 + a_1 r + a_0 = (r - r_1)(r - r_2) = r^2 - (r_1 + r_2)r + r_1 r_2$$
Then L factors the same way, as a composition of first order operators:
$$L = D^2 - (r_1 + r_2)D + r_1 r_2 I = (D - r_1 I) \circ (D - r_2 I).$$
(To check this is to work from right to left above:
$$(D - r_1 I) \circ (D - r_2 I) = D \circ (D - r_2 I) - r_1 I \circ (D - r_2 I)$$

$$= D \circ D - D \circ r_2 I - r_1 I \circ D - r_1 I \circ (-r_2 I)$$
$$= D^2 - r_2 D - r_1 D + r_1 r_2 I = L \,.)$$

Now, compute

$$(D - r_1 I)\left(e^{r_1 x}\right) = D\left(e^{r_1 x}\right) - r_1 e^{r_1 x} = r_1 e^{r_1 x} - r_1 e^{r_1 x} = 0.$$

Thus this factoring of L as

$$L = (D - r_1 I) \circ (D - r_2 I) = (D - r_2 I) \circ (D - r_1 I)$$

is (another reason) why $e^{r_1 x}, e^{r_2 x}$ both satisfy $L(y) = 0$ in case $r_1 \neq r_2$ are roots of the characteristic polynomial, since each exponential is transformed into the zero function by one of the linear factors of L. The case $r_1 = r_2$ is the double root case,

$$L = (D - r_1 I) \circ (D - r_1 I) = (D - r_1 I)^2 .$$

More generally, if the characteristic polynomial for the nth order homogeneous DE factors as

$$p(r) = (r - r_1)^{k_1} (r - r_2)^{k_2} ... (r - r_m)^{k_m}$$

then the corresponding operator L factors as a composition

$$L = (D - r_1 I)^{k_1} \circ (D - r_2 I)^{k_2} \circ ... \circ (D - r_m I)^{k_m}.$$

week 5.2
a) Let $f(x)$ be any differentiable function. Check that

$$(D - r_1 I)\left(f(x)e^{r_1 x}\right) = f'(x)e^{r_1 x} .$$

b) Deduce via successive application of the formula above, k times, that for $k \in \mathbb{N}$,

$$(D - r_1 I)^k f(x)e^{r_1 x} = f^{(k)}(x)e^{r_1 x} .$$

c) Use the result from **b** to explain why if L has a factor

$$(D - r_1 I)^{k_1}$$

with $k_1 > 1$, then

$$y(x) := \left(c_1 + c_2 x + ... + c_{k_1 - 1} x^{k_1 - 1} \right) e^{r_1 x} ,$$

i.e. any linear combination of the k_1 claimed solutions we wrote down in the algorithm for finding a solution space basis, does indeed satisfy $L(y) = 0$.

3.4: mechanical vibrations: setting up and solving differential equations and initial value problems for mechanical vibrations; underdamped, critically damped, and overdamped vibration problems; applications to pendulums, mass-spring configurations, and related problems; understanding and using phase-amplitude form for sinusoidal functions.

3.4: 3, **4**, 5, 6, **10**, (in **10** the weight of the buoy is $m\,g = \rho\,V g = \rho\,\pi\,r^2\,h\,g$. The text forgot to include the density ρ in computing the mass from the volume), 15, 17, 21.

week 5.3 Consider the unforced mass-spring-dashpot system with $m = 3$ and $k = 48$. In the following

problems the damping constant c will vary.
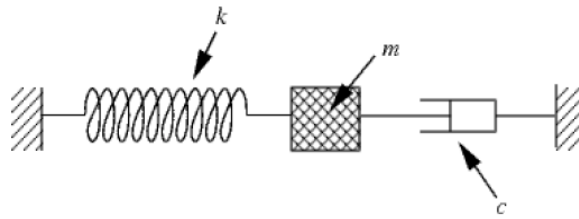
So, the displacement $x(t)$ of the mass from its equilibrium position will satsify the differential equation
$$3\,x''(t) + c\,x'(t) + 48\,x(t) = 0.$$
In each part below, we will consider the IVP with
$$x(0) = 3$$
$$x'(0) = -4.$$

a) Solve the IVP when there is no damping, i.e. $c = 0$. After finding your solution in linear combination form, convert into amplitude-phase form. Identify numerical values for amplitude, angular frequency, frequency, period, phase angle, and time delay.

b) Solve the IVP when $c = 1.2$ (we refer to the situation when c is small enough so that the characteristic polynomial ha complex roots as *underdamped*). Instructions: When finding roots of characteristic polynomials, round off to 5 digits.

c) Recall that a value of the damping constant induces so-called *critical damping* whenever the associated characteristic polynomial has a double real root. Find the value of c that leads to critical damping in this problem, and solve the IVP in this case.

d) Finally, solve the IVP when $c = 51$ (here, we are in the situation of *over-damping*).

e) Use Maple or other software (Matlab, Wolfram alpha, etc) to create a display containing the graphs of all four solutions above, on the interval $0 \le t \le 4$. Print out a copy, and label which graph corresponds to which solution.