Math 2280-001

Week 4 concepts and homework, due February 6.

Recall that problems which are not underlined are good for seeing if you can work with the underlying concepts; that the underlined problems are to be handed in; and that the Friday quiz will be drawn from all of these concepts and from these or related problems.

2.2: applications of population models.

week 4.1 Consider a bioreactor used by a yogurt factory to grow the bacteria needed to make yogurt. The growth of the bacteria is governed by the logistic equation

$$\frac{dP}{dt} = k \cdot P(M - P)$$

where P is the population in millions and t is the time in days. Recall that M is the carrying capacity of the reactor, and k is a constant that depends on the growth rate.

a) Through observation it is found that after a long time the population in the reactor stabilizes at 50 million bacteria, and that when the population of the reactor is 20 million bacteria the population increases at a rate of 12 million per day. From this, find k and M in the governing equation.

b) If the colony starts with a population of 10 million bacteria, how long will it take for the population to reach 80 % of carrying capacity?

c) Suppose the factory harvests the bacteria from the reactor once a week. The harvesting process takes a day, during which the reactor is not operational, leaving 6 days per week for the bacteria to grow in the reactor. The factory wants to maximize the amount of bacteria grown during these 6 days. To achieve this, $P'(t)$ should be at its maximum 3 days after harvesting. What initial population (after harvesting) gives the most growth over the 6-day period? What is the population change during this time?

d) Suppose the reactor is modified to allow for continual harvesting without shutting down the reactor. Let h be the rate at which the bacteria are harvested, in millions per day. Write down the new differential equation governing the bacteria population. What is the maximum rate of harvesting h that will not cause the population of bacteria to go extinct? (Harvesting at less than this rate will ensure that there is always a stable equilibrium point where P is positive.)

2.3: improved velocity-acceleration models:
constant, or constant plus linear drag forcing: **2, 3, 9, 10, 12**
quadratic drag: **13, 14, 17**
escape velocity: **25, 26**.

2.4-2.6: *numerical methods for approximating solutions to first order initial value problems.*
2.4: 4: *Euler's method*
2.5: 4: *improved Euler*
2.6: 4*: Runge-Kutta*

week 4.2) Runge-Kutta is based on Simpson's rule for numerical integration. Simpson's rule is based on the fact that for a subinterval of length $2h$, which by translation we may assume is the interval $-h \le x \le h$, the parabola $y = p(x)$ which passes through the points $(-h, y_0)$, $(0, y_1)$, (h, y_2) has integral

$$\int_{-h}^{h} p(x)\, dx = \frac{2h}{6} \cdot (y_0 + 4y_1 + y_2). \qquad (1)$$

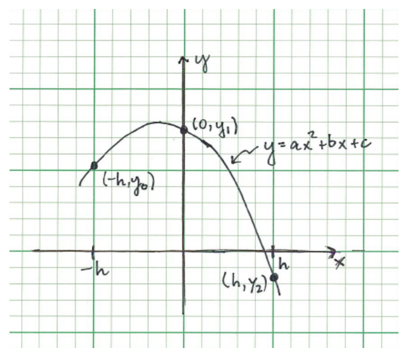
If we write the quadratic interpolant function $p(x)$ whose graph is this parabola as

$$p(x) = a x^2 + b x + c$$

with unknown parameters a, b, c then since we want $p(0) = y_1$ we solve $y_1 = p(0) = 0 + 0 + c$ to deduce that $c = y_1$.

a) Use the requirement that the graph of $p(x)$ is also to pass through the other two points, $(-h, y_0)$, (h, y_2) to express a, b in terms of h, y_0, y_1, y_2.

b) Compute $\int_{-h}^{h} p(x)\, dx$ for these values of a, b, c and verify equation (1) above.



Remark: If you've forgotten, or if you never talked about Simpson's rule in your Calculus class, here's how it goes: In order to approximate the definite integral of $f(x)$ on the interval $[a, b]$, you subdivide $[a, b]$ into $2n = N$ subintervals of width $\Delta x = \frac{b-a}{2n} = h$. Label the $x-$ values $x_0 = a, x_1 = a + h, x_2 = a + 2h, \ldots x_{2n} = b$, with corresponding $y-$values $y_i = f(x_i)$, $i = 0,\ldots n$. On each successive pair of intervals use the stencil above, estimating the integral of f by the integral of the parabola. This yields the very accurate (for large enough n) Simpson's rule formula

$$\int_a^b f(x)\, dx \approx \frac{2h}{6}\left((y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + \ldots + (y_{2n-2} + 4y_{2n-1} + y_{2n})\right),$$

i.e.

$$\int_a^b f(x)\, dx \approx \frac{b-a}{6n}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \ldots + 2y_{2n-2} + 4y_{2n-1} + y_{2n}).$$

http://en.wikipedia.org/wiki/Simpson's_rule

week 4.3) (Famous numbers revisited, section 2.6, page 135, of text). The mathy numbers e, $\ln(2)$, π can be well-approximated using approximate solutions to differential equations. We illustrate this on Wednesday Feb. 4 for e, which is $y(1)$ for the solution to the IVP

$$y'(x) = y$$
$$y(0) = 1.$$

Apply Runge-Kutta with $n = 10, 20, 40 \ldots$ subintervals, successively doubling the number of subintervals until you obtain the target number below - rounded to 9 decimal digits - twice in succession. We will do this in class for e, and you can modify that code if you wish.

a) $\ln(2)$ is $y(2)$, where $y(x)$ solves the IVP

$$y'(x) = \frac{1}{x}$$
$$y(1) = 0$$

(since $y(x) = \ln(x)$).

b) π is $y(1)$, where $y(x)$ solves the IVP

$$y'(x) = \frac{4}{x^2 + 1}$$
$$y(0) = 0$$

(since $y(x) = 4 \arctan(x)$).

Note that in **a,b** you are actually "just" using Simpson's rule from Calculus, since the right sides of these DE's only depend on the variable x and not on the value of the function $y(x)$. For reference:

```
> Digits := 16 : #how many digits to use in floating point numbers and calcululations
  evalf(e); #evaluate the floating point of e
  evalf(π);
```

2.718281828459045

3.141592653589793 **(1)**

```
>
```