Math 2280-001

Week 14-15 concepts and homework, due Wednesday April 29 at 5:00 p.m.

9.1-9.3: Fourier transform formulas: postponed problems from last week's homework:
9.3 Fourier sine and cosine series
19, 20
week 14.6) In **9.3.19** you successively antidifferentiate the sawtooth Fourier series (for $f(t) = t$ on $[-\pi, \pi]$ extended to be 2π-periodic) three times to find that the 2π-periodic extension of $f(t) = \frac{t^4}{24}$ has Fourier series

$$\frac{t^4}{24} = \frac{\pi^2 t^2}{12} - 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}\cos(n\,t) + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}\,.$$

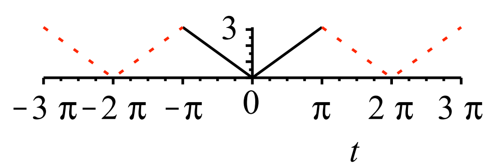
Compare your answer by having Maple find the Fourier coefficients of $\frac{t^4}{24}$ directly. Hint: The Maple commands you would need are on page 3 of Monday April 20 class notes. The two answers will not immediately look identical, although they are equivalent.

9.4: Understanding forced oscillation differential equations, with periodic forcing functions, via Fourier series expansions and superposition of particular solutions.
The following exercises are in the spirit of class notes/discussion on Friday April 24. Please make use of the particular solution table at the end of those notes.

week 15.1 Consider the forced oscillation problem

$$* \qquad x''(t) + x(t) = tent(t)$$

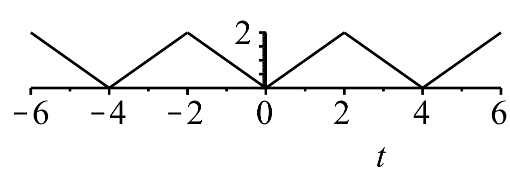
where $tent(t)$ is the 2π- periodic extension of the absolute value function, $|t|$ on the interval $[-\pi, \pi]$ to all of $\mathbb{R}$.



$$tent(t) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n\ odd}\frac{1}{n^2}\cos(n\,t)$$

We used convolution integrals on April 17 to show that solutions to this differential equation exhibit resonance. Use infinite superposition to find a particular solution to *, in order to explain why resonance occurs in this problem, and what the identify the term in the solution that is causing it.

week 15.2 Consider the re-scaled tent function with period 4, which on the interval $[-2, 2]$ is given by $f_2(t) = |t|$.



a) Find the Fourier series for f_2, either from scratch or by rescaling the one for $tent(t)$.

b) Find a particular solution to

$$** \qquad x''(t) + x(t) = f_2(t)$$

via infinite superposition. (Resonance cannot occur in this case because none of the Fourier terms for the forcing function f_2 will be oscillating with natural frequency $\omega_0 = 1$.)

week 15.3 Consider the slightly damped forced oscillation problem

$$* \qquad x''(t) + 0.02\, x'(t) + 1.01\, x(t) = tent(t)$$

Because there is damping, pure resonance will not occur. Because the damping coefficient is near zero, however, the steady periodic solution will potentially have large oscillations - if one of the sinusoidal functions in the Fourier expansion of $tent(t)$ has angular frequency ω near ω_0 - a manifestation of practical resonance.

a) Find $x_P(t) = x_{sp}(t)$, using infinite superposition, and identify the large-amplitude sinusoidal function which is causing practical resonance.

week 15.4 We've discussed the fact that for the undamped forced oscillation problem

$$x''(t) + \omega_0^2\, x(t) = f(t),$$

where $f(t)$ has period $P = 2\,L$, then resonance will only occur if one of the sinusoidal functions in the Fourier series for f,

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n \frac{\pi}{L} t\right) + \sum_{n=1}^{\infty} b_n \sin\left(n \frac{\pi}{L} t\right)$$

has $\omega_n = \dfrac{n\,\pi}{L} = \omega_0$ (and with at least one of a_n, b_n non-zero). Solving for the period $P = 2\,L$ of the forcing function, this means

$$2\,L = P = n \frac{2\,\pi}{\omega_0}.$$

In other words, the forcing function period is exactly n times the natural period for homogeneous solutions. An example of this occured in Example 4 of the Friday April 17 "resonance games", where the forcing function period was three times the natural period for homogeneous solutions. (We were pushing the swing every third time.) The forcing function had period $6\,\pi$ and was defined for $0 < t < 6\,\pi$ by

$$f_4(t) = \begin{cases} 1, & 0 < t < \pi \\ 0, & \pi < t < 6\,\pi \end{cases}$$

and the differential equation was

$$x''(t) + x(t) = f_4(t).$$

15.4a) Find the Fourier series for f_4.

15.4b) Check your Fourier coefficients using technology, as we've done several times in our class notes for different functions.

15.4c) Use infinite superposition to verify that resonance occurs in this problem, and identify which term in the particular solution sum is responsible for it.