Name....Solutions....

I.D. number....

Math 2280-001 Spring 2015
FINAL EXAM

This exam is closed-book and closed-note. You may use a scientific calculator, but not one which is capable of graphing or of solving differential or linear algebra equations. A Laplace Transform table and particular solution table are included with this exam. **In order to receive full or partial credit on any problem, you must show all of your work** and **justify your conclusions.** This exam counts for 30% of your course grade. It has been written so that there are 150 points possible, and the point values for each problem are indicated in the right-hand margin. **Good Luck!**

problem	score	possible
1	______	25
2	______	15
3	______	30
4	______	25
5	______	15
6	______	10
7	______	15
8	______	15
total	______	150

1a) Find the eigenvalues and eigenspace bases for the following matrix:

$$A = \begin{bmatrix} -3 & 1 \\ 2 & -4 \end{bmatrix}.$$

Hint: The characteristic polynomial has integer roots.

(8 points)

$$\begin{vmatrix} -3-\lambda & 1 \\ 2 & -4-\lambda \end{vmatrix} = \lambda^2 + 7\lambda + 12 - 2 = \lambda^2 + 7\lambda + 10$$
$$= (\lambda+2)(\lambda+5)$$

$$E_{\lambda=-2}: \quad \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 2 & -2 & 0 \end{array}\right] \Rightarrow \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$E_{\lambda=-5} \quad \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 2 & 1 & 0 \end{array}\right] \Rightarrow \vec{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

1b) Check that $A\,\underline{v} = \lambda\,\underline{v}$ for each eigenpair $(\lambda, \underline{v})$ that you found in part a. This is to catch any mistakes you may have made, since the matrix in A reappears frequently in this exam.

(2 points)

$$\begin{bmatrix} -3 & 1 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} = -2\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \checkmark$$

$$\begin{bmatrix} -3 & 1 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -5 \\ 10 \end{bmatrix} = -5\begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \checkmark$$

1c) Find e^{tA} for the matrix in part a. There are two approaches you may take: either use your work from part a to first find a fundamental matrix (also known as a non-singular Wronskian matrix) for the system $\underline{x}'(t) = A\underline{x}$, and work from there; or, use $AS = S\Lambda$ (in the form $A = S\Lambda S^{-1}$) to compute the power series for e^{tA} directly. If you successfully compute e^{tA} both ways you will receive 10 extra credit points.

(15 points)
(10 extra credit points also possible)

$\underline{\Phi(t)\Phi(0)^{-1}}$: soltn space basis to $\vec{x}' = A\vec{x}$ is $e^{-2t}\begin{bmatrix}1\\1\end{bmatrix}, e^{-5t}\begin{bmatrix}1\\-2\end{bmatrix}$

so may take $\Phi(t) = \begin{bmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t}\end{bmatrix}$

$$e^{tA} = \Phi(t)\Phi(0)^{-1} = \begin{bmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t}\end{bmatrix}\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -2\end{bmatrix}^{-1}}_{\frac{1}{-3}\begin{bmatrix}-2 & -1\\-1 & 1\end{bmatrix} = \frac{1}{3}\begin{bmatrix}2 & 1\\1 & -1\end{bmatrix}}$$

$$= \frac{1}{3}\begin{bmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t}\end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & -1\end{bmatrix} = \boxed{\frac{1}{3}\begin{bmatrix} 2e^{-2t}+e^{-5t} & e^{-2t}-e^{-5t} \\ 2e^{-2t}-2e^{-5t} & e^{-2t}+2e^{-5t}\end{bmatrix}}$$

$\underline{\text{diagonalization}}$

$$e^{At} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots$$
$$= I + tS\Lambda S^{-1} + \frac{t^2}{2!}S\Lambda^2S^{-1} + \frac{t^3}{3!}S\Lambda^3S^{-1} + \cdots$$
$$= S\left[I + t\Lambda + \frac{t^2}{2!}\Lambda^2 + \cdots\right]S^{-1}$$
$$e^{tA} = S e^{t\Lambda} S^{-1}$$

in our case $S = \begin{bmatrix} 1 & 1 \\ 1 & -2\end{bmatrix}$, $AS = \begin{bmatrix} -2 & -5 \\ -2 & 10\end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -2\end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & -5\end{bmatrix} = S\Lambda$

$$e^{tA} = \begin{bmatrix} 1 & 1 \\ 1 & -2\end{bmatrix} e^{t\Lambda} \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & -1\end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -2\end{bmatrix}\begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-5t}\end{bmatrix}\frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & -1\end{bmatrix}$$
$$= \frac{1}{3}\begin{bmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t}\end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & -1\end{bmatrix}$$

(as above)

= same final matrix as above.

2a) Use Laplace transform techniques to find the general solution to the undamped forced oscillator equation with resonance:

$$x''(t) + \omega_0^2 x(t) = \frac{F_0}{m}\cos(\omega_0 t)$$
$$x(0) = x_0$$
$$x'(0) = v_0 .$$

(10 points)

$$s^2X(s) - sx_0 - v_0 + \omega_0^2 X(s) = \frac{F_0}{m}\frac{s}{s^2+\omega_0^2}$$

$$X(s)(s^2+\omega_0^2) = \frac{F_0}{m}\frac{s}{s^2+\omega_0^2} + sx_0 + v_0$$

$$X(s) = \frac{F_0}{m}\frac{s}{(s^2+\omega_0^2)^2} + x_0\frac{s}{s^2+\omega_0^2} + v_0\frac{1}{s^2+\omega_0^2}$$

$$\boxed{x(t) = \frac{F_0}{m}\frac{t}{2\omega_0}\sin\omega_0 t + x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t}$$

2b) Consider the more general forced oscillation problem,

$$x''(t) + \omega_0^2 x(t) = f(t)$$
$$x(0) = x_0$$
$$x'(0) = v_0.$$

Find a formula for $x(t)$, that will be valid no matter what function $f(t)$ is used to force the system (as long as $f(t)$ is piecewise continuous with at most exponential growth, so that it has a Laplace transform). Hint: part of your solution will be a convolution integral involving the forcing function f.

(5 points)

$$\mathcal{L}: \quad s^2X(s) - sx_0 - v_0 + \omega_0^2X(s) = F(s)$$

$$X(s)(s^2+\omega_0^2) = F(s) + x_0\frac{s}{s^2+\omega_0^2} + \frac{v_0}{s^2+\omega_0^2}$$

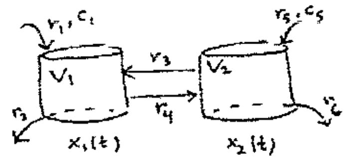
$$X(s) = F(s)\underbrace{\frac{1}{s^2+\omega_0^2}}_{G(s)} + \cancel{x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin}\ x_0\frac{s}{s^2+\omega_0^2} + \frac{v_0}{\omega_0}\frac{\omega_0}{s^2+\omega_0^2}$$

$$g(t) = \frac{1}{\omega_0}\sin\omega_0 t$$

$$x(t) = f*g(t) + x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t$$

$$\boxed{x(t) = \int_0^t f(\tau)\frac{1}{\omega_0}\sin(\omega_0(t-\tau))\,d\tau + x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t}$$

3) Consider a general input-output model with two compartments as indicated below. The compartments contain volumes V_1, V_2 and solute amounts $x_1(t), x_2(t)$ respectively. The flow rates (volume per time) are indicated by r_i, $i = 1..6$. The two input concentrations (solute amount per volume) are c_1, c_5.



3a) What is the system of 4 first order differential equations governing the volumes $V_1(t)$, $V_2(t)$ and solute amounts $x_1(t)$, $x_2(t)$?

(8 points)

$$V_1'(t) = r_1 + r_3 - r_2 - r_4$$
$$V_2'(t) = r_4 + r_5 - r_3 - r_6$$
$$x_1' = r_1 c_1 + r_3 \frac{x_2}{V_2} - (r_2 + r_4)\frac{x_1}{V_1}$$
$$x_2' = r_5 c_5 + r_4 \frac{x_1}{V_1} - (r_3 + r_6)\frac{x_2}{V_2}$$

3b) Suppose $r_2 = r_3 = 100$, $r_1 = r_4 = r_5 = 200$, $r_6 = 300 \ \frac{gal}{hour}$. Verify from your work in 1a that the volumes $V_1(t)$, $V_2(t)$ remain constant.

(2 points)

$V_1' = 200 + 100 - 100 - 200 = 0$ so $V_1 = $ const
$V_2' = 200 + 200 - 100 - 300 = 0$ so $V_2 = $ const

3c) Using the flow rates above, $c_1 = 0.05$, $c_5 = 0.3 \ \frac{lb}{gal}$, $V_1 = V_2 = 100 \ gal$, show that the amounts of solute $x_1(t)$ in tank 1 and $x_2(t)$ in tank 2 satisfy

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 10 \\ 60 \end{bmatrix}.$$

(5 points)

$$x_1' = r_1 c_1 + r_3 \frac{x_2}{V_2} - (r_2 + r_4)\frac{x_1}{V_1}$$
$$= 200(.05) + 100\frac{x_2}{100} - (300)\frac{x_1}{100} = -3x_1 + x_2 + 10 \ \checkmark$$

$$x_2' = r_5 c_5 + r_4 \frac{x_1}{V_1} - (r_3 + r_6)\frac{x_2}{V_2}$$
$$= 200(.3) + 200\frac{x_1}{100} - 400\frac{x_2}{100} = 2x_1 - 4x_2 + 60 \ \checkmark$$

3d) Find the general solution to

$$* \qquad \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 10 \\ 60 \end{bmatrix}.$$

Note that the matrix in this problem is the same as the one in problem 1. You may refer to your results from that problem. Hint: You may use $\underline{x} = \underline{x}_P + \underline{x}_{HP}$ Laplace transforms, or any other method we've discussed in this course, in order to find the general solution.

(10 points)

from #1. $\vec{x}_H(t) = c_1 e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-5t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

for $\vec{x}_p$ try $\vec{x}_p = \vec{c}$ const. to satisfy *

$$\vec{x}_p' = \vec{c}\,' = \vec{0} = \begin{bmatrix} -3 & 1 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} 10 \\ 60 \end{bmatrix}$$

$$-3c_1 + c_2 + 10 = 0$$
$$+2c_1 - 4c_2 + 60 = 0$$

$$-3c_1 + c_2 = -10$$
$$c_1 - 2c_2 = -30$$

$2E_1 + E_2 \Rightarrow -5c_1 = -50 \Rightarrow c_1 = 10 \Rightarrow c_2 = 20$

$$\vec{x}_p = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$$

$$\boxed{\vec{x}(t) = \begin{bmatrix} 10 \\ 20 \end{bmatrix} + c_1 e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-5t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}}$$

3e) Use your work from d to solve the initial value problem

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 10 \\ 60 \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 20 \\ 0 \end{bmatrix}.$$

(5 points)

@ $t = 0$

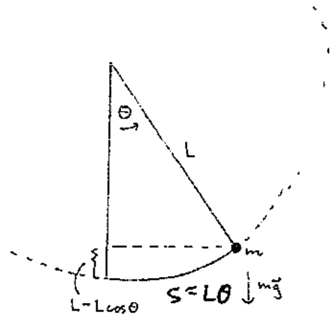
$$\begin{bmatrix} 20 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$10 = c_1 + c_2$$
$$-20 = c_1 - 2c_2$$

$E_1 - E_2 \Rightarrow 30 = 3c_2 \Rightarrow c_2 = 10 \Rightarrow c_1 = 0.$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix} + 10e^{-5t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

4) Although we usually use a mass-spring configuration to give context for studying second order differential equations, the rigid-rod pendulum also effectively exhibits several key ideas from this course. Recall that in the undamped version of this configuration, we let the pendulum rod length be L, assume the rod is massless, and that there is a mass m attached at the end on which the vertical graviational force acts with force $m \cdot g$. This mass will swing in a circular arc of signed arclength $s = L \cdot \theta$ from the vertical, where θ is the angle in radians from vertical. The configuration is indicated below.



4a) Use the fact that the undamped system is conservative, to derive the differential equation for $\theta(t)$,

$$\theta''(t) + \frac{g}{L} \cdot \sin(\theta(t)) = 0.$$

(10 points)

Hint: Begin by express the TE=KE+PE in terms of the function $\theta(t)$ and its derivatives. Then compute $TE'(t)$ and set it equal to zero.

$$TE = KE + PE = \frac{1}{2} m v^2 + mgh$$

$$v = \frac{ds}{dt} = L\theta'(t)$$

$$h = L - L\cos\theta$$

$$TE = \frac{1}{2} m L^2 \theta'(t)^2 + mg\,(L - L\cos\theta(t))$$

$$0 = TE'(t) = \cancel{\tfrac{1}{2}} m L^2 \cancel{2} \theta' \theta'' + mg\,(-L)(-\sin\theta(t))\,\theta'(t)$$

$$0 = mL\theta' \left[L\theta'' + g\sin\theta \right]$$

$\uparrow$

$\neq 0$ except at isolated points

$$\Rightarrow \quad 0 = L\theta'' + g\sin\theta$$

$$\left(\text{or } 0 = \theta'' + \frac{g}{L}\sin\theta\right)$$

4b) For small oscillations $(\theta(t) \approx 0)$ we replaced the non-linear differential equation in a with the linearization

$$(1) \qquad \theta''(t) + \frac{g}{L}\theta(t) = 0.$$

Use the Taylor series for $\sin(\theta)$ to explain why this is a good approximation to the exact differential equation in a, when e.g. $|\theta| < 0.1$ (radians).

(5 points)

$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$ is an alternating series

for an alternating series the error between a partial sum and the entire sum is bounded by the absolute value of the next term.

Thus, if $|\theta| < .1$,

$$|\sin\theta - \theta| < \frac{.1^3}{3!} = \frac{.001}{6} < .0002.$$

4c) Carefully describe the connection between solutions $\theta(t)$ to the second order linear differential equation in b and solutions $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ to the first order system of differential equations

$$(2) \qquad \begin{aligned} x_1'(t) &= x_2 \\ x_2'(t) &= -\frac{g}{L}x_1. \end{aligned}$$

(4 points)

If $\theta(t)$ solves (1) define

$$x_1(t) = \theta(t)$$
$$x_2(t) = \theta'(t)$$

then $x_1' = x_2$

$$x_2' = x_1'' = \theta'' = -\frac{g}{L}\theta = -\frac{g}{L}x_1$$

So $\begin{bmatrix} \theta \\ \theta' \end{bmatrix}$ solves (2)

(Conversely, if $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ solves (2), then $x_1'' = x_2' = -\frac{g}{L}x_1$

i.e. $x_1'' + \frac{g}{L}x_1 = 0$

So x_1 solves (1).)

4d) In case the numerical value of $\frac{g}{L} = 1$ the differential equation in b becomes

$$\theta''(t) + \theta(t) = 0$$

and the corresponding first order system in c becomes

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}$$

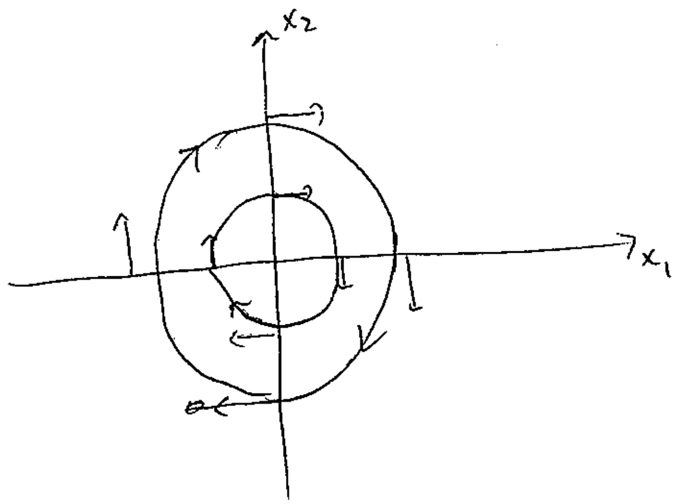
Sketch the phase portrait for the first order system, and classify the origin as one of: nodal source, nodal sink, saddle point, spiral source, spiral sink, stable center.

(6 points)

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = (\lambda + i)(\lambda - i)$$

so origin is a stable center

pt.	tang
$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$	$\begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}$.
$c\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$c\begin{bmatrix} 0 \\ -1 \end{bmatrix}$
$c\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$c\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.



in fact, one can find the soltns, since the $\theta(t)$ solving $\theta'' + \theta = 0$ are $\theta(t) = C\cos(t-\alpha)$

and $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \theta' \end{bmatrix} = \begin{bmatrix} C\cos(t-\alpha) \\ -C\sin(t-\alpha) \end{bmatrix}$

are the soltns to the system. These parametric curves are circles of radius C, traversing clockwise around the origin

5a) Find the general solution to the system of differential equations

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

(8 points)

Hint: This second order system of DE's could be modeling a two-mass, three-spring system without damping and so it will have solutions that oscillate. Also, this is the same matrix as in problem 1 and you may use results from that problem.

from #1, $\lambda = -2, -5$

$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\omega = \sqrt{-\lambda}$ so

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = (c_1 \cos\sqrt{2}t + c_2 \sin\sqrt{2}t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (c_3 \cos\sqrt{5}t + c_4 \sin\sqrt{5}t) \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

5b) Identify and describe the two fundamental modes of oscillation in the system above.

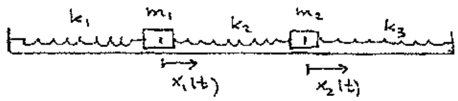
(2 points)

slower, in phase mode, with $\omega = \sqrt{2}$, each mass oscillates with same amplitude

faster, out of phase mode – mass 2 is 180° out of phase with mass 1 & with twice the amplitude, $\omega = \sqrt{5}$

5c) Set $m_1 = 1$, $m_2 = \frac{1}{2}$, $k_1 = 2$, $k_2 = 1$, $k_3 = 1$. Show that the displacements $x_1(t), x_2(t)$ of the two masses from equilibrium in the configuration below satisfy the system in part 5a, i.e.

$$x_1''(t) = -3x_1 + x_2$$
$$x_2''(t) = 2x_1 - 4x_2$$



Hint: Use Newton's second law that mass times acceleration equals net forces.

(5 points)

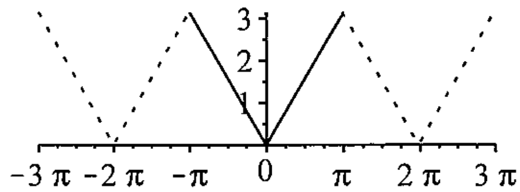
$$m_1 x_1'' = -k_1 x_1 + k_2(x_2 - x_1)$$
$$m_2 x_2'' = -k_2(x_2 - x_1) - k_3 x_2$$

$$1 \cdot x_1'' = -2x_1 + (x_2 - x_1) = -3x_1 + x_2$$
$$\frac{1}{2} x_2'' = -(x_2 - x_1) - x_2 = x_1 - 2x_2$$

$$\Rightarrow \quad x_1'' = -3x_1 + x_2$$
$$x_2'' = 2x_1 - 4x_2$$

6) We consider a 2π-periodic tent-wave function, given on the interval $(-\pi, \pi)$ by $f(t) = |t|$.

Here's a piece of the graph of this function.



Derive the Fourier series for $f(t)$,

$$f(t) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n\ odd}\frac{1}{n^2}\cos(n\,t)$$

(10 points)

$f(t)$ is even, so $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt = 0.$

$\frac{a_0}{2}$ = average value of f $= \frac{\pi}{2}$;
but one can also compute

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}|t|\,dt = \frac{2}{\pi}\int_0^{\pi} t\,dt = \frac{2}{\pi}\left.\frac{t^2}{2}\right]_0^{\pi} = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{|t|\cos nt}_{\text{even}}\,dt$$

$$= \frac{2}{\pi}\int_0^{\pi}\underbrace{t}_{u}\,\underbrace{\cos nt\,dt}_{dv}$$

$$du = dt \qquad v = +\frac{\sin nt}{n}$$

$$= \frac{2}{\pi}\left[\underbrace{\left. t\left(+\cancel{\frac{\sin nt}{n}}\right)\right]_0^{\pi}}_{0} - \int_0^{\pi} +\frac{\sin nt}{n}\,dt\right]$$

$$= \frac{2}{\pi}\underbrace{\left[\frac{\cos nt}{n^2}\right]_0^{\pi}}_{\frac{1}{n^2}(\cos n\pi - 1)} = \begin{cases} 0 & n \text{ even} \\ -\frac{4}{\pi n^2} & n \text{ odd}. \end{cases}$$

So
$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nt + \sum_{n=1}^{\infty} b_n\sin nt$$

$$\boxed{= \frac{\pi}{2} - \frac{4}{\pi}\sum_{\substack{n=1\\ n\ odd}}^{\infty}\frac{1}{n^2}\cos nt}$$

7) Consider the tent-wave function $f(t)$ from problem 6, and the forced oscillation problem

$$x''(t) + 9\cdot x(t) = f(t).$$

7a) Discuss whether or not resonance occurs.

(5 points)

$\omega_0 = 3$, so the $n=3$ term $-\frac{4}{\pi}\frac{1}{9}\cos 3t$ of $f(t)$ will cause resonance

7b) Find the general solution for this forced oscillation problem. Hint: Use the Fourier series for $f(t)$ given in problem 6. You may make use of the particular solutions table at the end of the exam.

(10 points)

$x'' + 9x = \frac{\pi}{2}$

has $x_p = \frac{\pi}{18}$

for $n \neq 3$, x_p for

$x'' + 9x = -\frac{4}{\pi}\frac{1}{n^2}\cos nt$

is $x_p = -\frac{4}{\pi}\frac{1}{n^2}\frac{1}{9-n^2}\cos nt$

for $n=3$, x_p for

$x'' + 9x = -\frac{4}{\pi}\frac{1}{9}\cos 3t$

is $x_p = -\frac{4}{\pi}\frac{1}{9}\frac{t}{6}\sin 3t$

So, for $x'' + 9x = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n \text{ odd}} \frac{1}{n^2}\cos nt$

$$\boxed{\begin{aligned} x &= x_H + x_p \\ x &= c_1\cos 3t + c_2 \sin 3t + \frac{\pi}{18} - \frac{2}{27\pi} t\sin 3t - \frac{4}{\pi}\sum_{\substack{n \text{ odd} \\ n\neq 3}} \frac{1}{n^2}\frac{1}{9-n^2}\cos nt \end{aligned}}$$

8) General principles: Pick 3 of the following 4 parts to solve. You will receive credit for the best 3 solutions, for a possible total score of 15 points.

8a) Suppose that $\underline{v}$ is an eigenvector of A, with eigenvalue λ. Verify that $\underline{x}(t) = e^{\lambda t}\underline{v}$ is a solution to

$$\underline{x}'(t) = A\,\underline{x}.$$

(5 points)

for $\vec{x}(t) = e^{\lambda t}\vec{v}$

$\vec{x}'(t) = \lambda e^{\lambda t}\vec{v}$

$A\vec{x} = Ae^{\lambda t}\vec{v} = e^{\lambda t}A\vec{v} = e^{\lambda t}\lambda\vec{v}$

thus $\vec{x}' = A\vec{x} \quad (= \lambda e^{\lambda t}\vec{v})$

8b) Suppose that $\underline{v}$ is an eigenvector of A, with eigenvalue λ. Suppose $\lambda < 0$ and define $\omega = \sqrt{-\lambda}$. Verify that $\underline{x}(t) = \cos(\omega t)\underline{v}$ and $\underline{y}(t) = \sin(\omega t)\underline{v}$ are solutions to

$$\underline{x}''(t) = A\,\underline{x}$$

(5 points)

for $\vec{x} = \cos\omega t\,\vec{v}$

$\vec{x}' = -\omega\sin\omega t\,\vec{v}$

$\vec{x}'' = -\omega^2\cos\omega t\,\vec{v}$

& $A\vec{x} = A\cos\omega t\,\vec{v}$

$= \cos\omega t\,A\vec{v}$

$= \cos\omega t\,\lambda\vec{v}$

since $\omega^2 = -\lambda$

$-\omega^2 = \lambda$

and $\vec{x}'' = A\vec{x}$

the argument for $\vec{x} = \sin\omega t\,\vec{v}$ is identical.

8c) Prove that if $L : V \rightarrow W$ is a linear transformation, and if $y_P \in V$ solves the nonhomogeneous equation

$$L(y_P) = f$$

then <u>every</u> solution of the equation

$$L(y) = f$$

is of the form $y = y_P + y_H$, where y_H is some solution of the homogeneous equation

$$L(y) = 0.$$

(5 points)

1. If $L(y_p) = f$
and $L(y_H) = 0$
then $L(y_p + y_H) = L(y_p) + L(y_H)$ (L linear)
$= f + 0$
$= f$
So $y_p + y_H$ is also a soln to $L(y) = f$.

2. If y is <u>any</u> soln to $L(y) = f$
then $y = y_p + (y - y_p)$, and, applying L, $L(y) = L(y_p) + L(y - y_p)$
$f = f + L(y - y_p) \Rightarrow L(y - y_p) =$
so $y - y_p$ is a y_H.

8d) We discussed the analogy between constant coefficient first-order linear differential equations (in Chapter 1), and first order systems of differential equations (In Chapter 5). Use matrix exponentials and the "integrating factor" technique to show that for first order systems with constant matrix A, the general solution to

$$\underline{x}'(t) = A\underline{x} + \underline{f}(t)$$

is given by the formula

$$\underline{x}(t) = e^{tA}\left(\int e^{-tA}\underline{f}(t)\, dt\right) + e^{tA}\underline{c}.$$

(In the formula above, $\int e^{-tA}\underline{f}(t)\, dt$ is standing for any particular antiderivative of $e^{-tA}\underline{f}(t)$, and the displayed formula is expressing $\underline{x}(t)$ as $\underline{x}_P + \underline{x}_H$)

<u>Hint</u>: begin by rewriting the system as

$$\underline{x}'(t) - A\underline{x} = \underline{f}(t)$$

and then find an appropriate (matrix) integrating factor.

(5 points)

$$\vec{x}' - A\vec{x} = \vec{f}$$

$$e^{-At}(\vec{x}' - A\vec{x}) = e^{-At}\vec{f}$$

$$\frac{d}{dt}\left(e^{-At}\vec{x}\right) = e^{-At}\vec{f}$$

(since $\frac{d}{dt}\left(e^{-At}\vec{x}\right) = e^{-At}(-A)\vec{x} + e^{-At}\vec{x}'$ = left side)

integrate

$$e^{-At}\vec{x}(t) = \int e^{-At}\vec{f}(t)\,dt + \vec{c}$$

$e^{At}e^{-At} = I$

$$\Rightarrow \quad \vec{x}(t) = e^{At}\left(\int e^{-At}\vec{f}(t)\,dt + \vec{c}\right)$$

$$\vec{x}(t) = e^{tA}\int e^{-tA}\vec{f}(t)\,dt + e^{tA}\vec{c}$$

Fourier series information: For $f(t)$ of period $P = 2L$,

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n\frac{\pi}{L}t\right) + \sum_{n=1}^{\infty} b_n \sin\left(n\frac{\pi}{L}t\right)$$

with

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\,dt \qquad \left(\text{so } \frac{a_0}{2} = \frac{1}{2L}\int_{-L}^{L} f(t)\,dt \text{ is the average value of } f\right)$$

$$a_n := \left\langle f, \cos\left(n\frac{\pi}{L}t\right)\right\rangle = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(n\frac{\pi}{L}t\right) dt, \quad n \in \mathbb{N}$$

$$b_n := \left\langle f, \sin\left(n\frac{\pi}{L}t\right)\right\rangle = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(n\frac{\pi}{L}t\right) dt, \quad n \in \mathbb{N}$$

Particular solutions from Chapter 3 or Laplace transform table:

$$x''(t) + \omega_0^2 x(t) = A\,sin(\omega t)$$

$$x_p(t) = \frac{A}{\omega_0^2 - \omega^2}\sin(\omega t) \qquad \text{when } \omega \neq \omega_0$$

$$x_p(t) = -\frac{t}{2\omega_0} A\cos(\omega_0 t) \qquad \text{when } \omega = \omega_0$$

....

$$x''(t) + \omega_0^2 x(t) = A\cos(\omega t)$$

$$x_p(t) = \frac{A}{\omega_0^2 - \omega^2}\cos(\omega t) \qquad \text{when } \omega \neq \omega_0$$

$$x_p(t) = \frac{t}{2\omega_0} A\sin(\omega_0 t) \qquad \text{when } \omega = \omega_0$$

....

$$x'' + c\,x' + \omega_0^2 x = A\cos(\omega t) \qquad c > 0$$

$$x_p(t) = x_{sp}(t) = C\cos(\omega t - \alpha)$$

with

$$C = \frac{A}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + c^2\omega^2}}\ .$$

$$\cos(\alpha) = \frac{\omega_0^2 - \omega^2}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + c^2\omega^2}}$$

$$\sin(\alpha) = \frac{c\,\omega}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + c^2\omega^2}}\ .$$

....

$$x'' + c\,x' + \omega_0^2 x = A\sin(\omega t) \qquad c > 0$$

$$x_p(t) = x_{sp}(t) = C\sin(\omega t - \alpha)$$

with

$$C = \frac{A}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + c^2\omega^2}}\ .$$

$$\cos(\alpha) = \frac{\omega^2 - \omega_0^2}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + c^2\omega^2}}$$

$$\sin(\alpha) = \frac{c\,\omega}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + c^2\omega^2}}$$

Table of Laplace Transforms

This table summarizes the general properties of Laplace transforms and the Laplace transforms of particular functions derived in Chapter 10.

Function	Transform	Function	Transform
$f(t)$	$F(s)$	e^{at}	$\dfrac{1}{s-a}$
$af(t)+bg(t)$	$aF(s)+bG(s)$	$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$
$f'(t)$	$sF(s)-f(0)$	$\cos kt$	$\dfrac{s}{s^2+k^2}$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$	$\sin kt$	$\dfrac{k}{s^2+k^2}$
$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$	$\cosh kt$	$\dfrac{s}{s^2-k^2}$
$\int_0^t f(\tau)\,d\tau$	$\dfrac{F(s)}{s}$	$\sinh kt$	$\dfrac{k}{s^2-k^2}$
$e^{at}f(t)$	$F(s-a)$	$e^{at}\cos kt$	$\dfrac{s-a}{(s-a)^2+k^2}$
$u(t-a)f(t-a)$	$e^{-as}F(s)$	$e^{at}\sin kt$	$\dfrac{k}{(s-a)^2+k^2}$
$\int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$	$\dfrac{1}{2k^3}(\sin kt-kt\cos kt)$	$\dfrac{1}{(s^2+k^2)^2}$
$tf(t)$	$-F'(s)$	$\dfrac{t}{2k}\sin kt$	$\dfrac{s}{(s^2+k^2)^2}$
$t^nf(t)$	$(-1)^nF^{(n)}(s)$	$\dfrac{1}{2k}(\sin kt+kt\cos kt)$	$\dfrac{s^2}{(s^2+k^2)^2}$
$\dfrac{f(t)}{t}$	$\int_s^\infty F(\sigma)\,d\sigma$	$u(t-a)$	$\dfrac{e^{-as}}{s}$
$f(t)$, period p	$\dfrac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)\,dt$	$\delta(t-a)$	e^{-as}
1	$\dfrac{1}{s}$	$(-1)^{[\![t/a]\!]}$ (square wave)	$\dfrac{1}{s}\tanh\dfrac{as}{2}$
t	$\dfrac{1}{s^2}$	$\left[\!\left[\dfrac{t}{a}\right]\!\right]$ (staircase)	$\dfrac{e^{-as}}{s(1-e^{-as})}$
t^n	$\dfrac{n!}{s^{n+1}}$		
$\dfrac{1}{\sqrt{\pi t}}$	$\dfrac{1}{\sqrt{s}}$		
t^a	$\dfrac{\Gamma(a+1)}{s^{a+1}}$		