Name......Solutions..

I.D. number..

Math 2280-001
Second Midterm Exam
April 3, 2015

This exam is closed-book and closed-note. You may use a scientific calculator, but not one which is capable of graphing or of solving differential or linear algebra equations. **In order to receive full or partial credit on any problem, you must show all of your work** and **justify your conclusions.** There are 100 points possible, and the point values for each problem are indicated in the right-hand margin.
Good Luck!

Score		possible
1		20
2		25
3		30
4		5
5		20
total		100

1) Consider the following initial value problem, which could arise from Newton's second law in a forced mass-spring oscillation problem:

$$x''(t) + 2x'(t) + 10x(t) = 26\sin(2t)$$
$$x(0) = 0$$
$$x'(0) = 6.$$

1a) As a step in solving this IVP, find the general solution to the (underdamped) homogenous differential equation

$$x''(t) + 2x'(t) + 10x(t) = 0$$

(5 points)

$p(r) = r^2 + 2r + 10$
$= (r+1)^2 + 9$
$= (r+1+3i)(r+1-3i)$

roots $r = -1 \pm 3i$

$x_H(t) = c_1 e^{-t}\cos 3t + c_2 e^{-t}\sin 3t$

1b) Find a particular solution to

$$x''(t) + 2x'(t) + 10x(t) = 26\sin(2t)$$

using the method of undetermined coefficients.

(10 points)

$+10\,(x_p = A\cos 2t + B\sin 2t)$
$+2\,(x_p' = -2A\sin 2t + 2B\cos 2t)$
$+1\,(x_p'' = -4A\cos 2t - 4B\sin 2t)$

$L(x_p) = \cos 2t\,(10A + 4B - 4A)$
$+ \sin 2t\,(10B - 4A - 4B)$
$= \cos(2t)(6A + 4B)$
$+ \sin(2t)(-4A + 6B)$

want $= 26\sin 2t$ so $\begin{matrix} 6A + 4B = 0 \\ -4A + 6B = 26 \end{matrix} \Rightarrow \begin{matrix} 3A + 2B = 0 \\ -2A + 3B = 13 \end{matrix}$

$\Rightarrow \begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 13 \end{bmatrix} \Rightarrow \begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{13}\begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 13 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}.$ $\boxed{x_p = -2\cos 2t + 3\sin 2t}$

1c) Use your work from parts a,b to solve the initial value problem at the top of this page.

(5 points)

$x = x_p + x_H$

$x = -2\cos 2t + 3\sin 2t + c_1 e^{-t}\cos 3t + c_2 e^{-t}\sin 3t$

$x(0) = 0 = -2 + c_1 \Rightarrow c_1 = 2$

$x'(0) = 6 = 6 - c_1 + 3c_2 \Rightarrow c_1 = 3c_2 \Rightarrow c_2 = \frac{1}{3}c_1 = \frac{2}{3}$

$\boxed{x(t) = -2\cos 2t + 3\sin 2t + 2e^{-t}\cos 3t + \frac{2}{3}e^{-t}\sin 3t}$

2a) The homogeneous differential equation that you solved in 1a, namely

$$(1)\quad x''(t) + 2x'(t) + 10x(t) = 0$$

corresponds to the following first order system of two differential equations:

$$(2)\quad \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -10 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Carefully explain the correspondence between solutions $x(t)$ to the second order differential equation and solutions $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ to the first order system.

(5 points)

If $x(t)$ solves (1)
let $x_1(t) := x(t)$
$x_2(t) := x'(t)$
then $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ solves (2):

then
$$\left.\begin{aligned} x_1' &= x' = x_2 \\ x_2' &= x'' = -10x - 2x' \\ x_2' &= -10x_1 - 2x_2 \end{aligned}\right\} \text{ which is (2).}$$

conversely, if $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ solves (2), let $x(t) := x_1(t)$.
Then $x(t)$ solves (1):

$$x' = x_1' = x_2$$
$$x'' = x_2' = -10x_1 - 2x_2$$
$$x'' = -10x - 2x'$$
$$x'' + 2x' + 10x = 0$$

2b) Use your work from 1a) to write down the general (homogenous) solution to the first order system

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -10 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Hint: It would be ugly to solve this problem using the eigenvalue-eigenvector methods of Chapter 5.

(5 points)

use $\begin{bmatrix} x(t) \\ x'(t) \end{bmatrix}$ from 1a).

$$x_1 = x(t) = c_1 e^{-t}\cos 3t + c_2 e^{-t}\sin 3t$$
$$x_2 = x'(t) = c_1\left(-e^{-t}\cos 3t + 3e^{-t}\sin 3t\right) + c_2\left(-e^{-t}\sin 3t + 3e^{-t}\cos 3t\right)$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 e^{-t}\begin{bmatrix} \cos 3t \\ -\cos 3t + 3\sin 3t \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} \sin 3t \\ -\sin 3t + 3\cos 3t \end{bmatrix}$$

2c) Find the (complex) eigenvalues for the matrix in 2b. Use this information to classify the phase portrait as one of: nodal source, nodal sink, saddle, spiral source, spiral sink, center. For your convenience, the system is repeated here:

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -10 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

(5 points)

$$\begin{vmatrix} -\lambda & 1 \\ -10 & -2-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 10$$
$$= (\lambda+1)^2 + 9 = 0$$
$$(\lambda+1)^2 = -9$$
$$\lambda + 1 = \pm 3i$$
$$\lambda = -1 \pm 3i$$

λ complex, $Re(\lambda) < 0$.

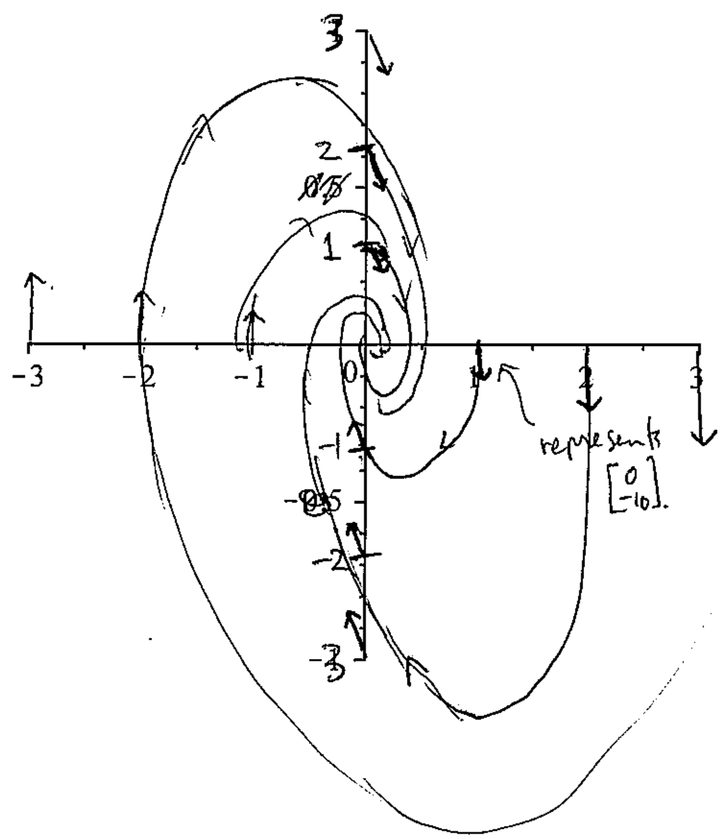
spiral sink

2d) Sketch the phase diagram for the system in 2b. Hint: It is possible to construct an adequate picture by using the classification in 2c and by checking the tangent vector field in several places.

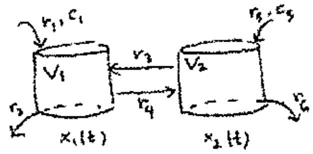
(10 points)

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -10 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

pt	tang vect
$c\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$c\begin{bmatrix} 0 \\ -10 \end{bmatrix}$
$c\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$c\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.



3) Consider a general input-output model with two compartments as indicated below. The compartments contain volumes V_1, V_2 and solute amounts $x_1(t), x_2(t)$ respectively. The flow rates (volume per time) are indicated by r_i, $i = 1..6$. The two input concentrations (solute amount per volume) are c_1, c_5.



3a) Suppose $r_2 = r_3 = r_6 = 100$, $r_1 = r_4 = 200$, $r_5 = 0$ $\dfrac{gal}{hour}$. Explain why the volumes $V_1(t), V_2(t)$ remain constant.

(4 points)

$$V_1' = r_1 + r_3 - r_2 - r_4 = 200 + 100 - 100 - 200 = 0$$

$$V_2' = r_5 + r_4 - r_3 - r_6 = 0 + 200 - 100 - 100 = 0$$

so V_1, V_2 are constant

3b) Using the flow rates above, $c_1 = 0.6$, $c_5 = 0$ $\dfrac{lb}{gal}$, $V_1 = V_2 = 100\ gal$, show that the amounts of solute $x_1(t)$ in tank 1 and $x_2(t)$ in tank 2 satisfy

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 120 \\ 0 \end{bmatrix}.$$

(6 points)

$$x_1' = r_1 c_1 - r_2 \frac{x_1}{V_1} + r_3 \frac{x_2}{V_2} - r_4 \frac{x_1}{V}$$

$$= 200(.6) - 100 \frac{x_1}{100} + 100 \frac{x_2}{100} - 200 \frac{x_1}{100}$$

$$= 120 - x_1 + x_2 - 2x_1$$

$$\boxed{x_1' = -3x_1 + x_2 + 120}$$

$$x_2' = r_5 c_5 + r_4 \frac{x_1}{V_1} - (r_3 + r_6) \frac{x_2}{V_2}$$

$$= 0 + 200 \frac{x_1}{100} - 200 \frac{x_2}{100}$$

$$\boxed{x_2' = 2x_1 - 2x_2}$$

3c) Find the general solution to the system

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 120 \\ 0 \end{bmatrix},$$

Hint: As steps you will want to find the homogeneous solution $\underline{x}_H$ and then a particular solution which is a constant vector, $\underline{x}_p = \underline{c}$.

(15 points)

$\vec{x}_H$: $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$|A - \lambda I| = \begin{vmatrix} -3-\lambda & 1 \\ 2 & -2-\lambda \end{vmatrix}$$

$$= \lambda^2 + 5\lambda + 4$$
$$= (\lambda+4)(\lambda+1)$$

$\lambda = -4$:

$$\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 2 & 0 \end{array}$$

$\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\vec{x} = e^{-4t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda = -1$

$$\begin{array}{cc|c} -2 & 1 & 0 \\ 2 & -1 & 0 \end{array}$$

$\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$e^{-t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\vec{x}_H(t) = c_1 e^{-4t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

$\vec{x}_p = \vec{c}$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} 120 \\ 0 \end{bmatrix}.$$

$0 = -3c_1 + c_2 + 120$

$0 = 2c_1 - 2c_2 \Rightarrow c_1 = c_2$

$0 = -2c_1 + 120 \Rightarrow c_1 = 60$, $c_2 = 60$

$$\vec{x}_p = \begin{bmatrix} 60 \\ 60 \end{bmatrix}$$

$\vec{x} = \vec{x}_p + \vec{x}_H$

$$= \begin{bmatrix} 60 \\ 60 \end{bmatrix} + c_1 e^{-4t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

3d) Solve the initial value problem for 3c, assuming there is initially no solute in either tank.

(5 points)

$$\vec{x}(t) = \begin{bmatrix} 60 \\ 60 \end{bmatrix} + c_1 e^{-4t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

@ $t = 0$.

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 60 \\ 60 \end{bmatrix} + c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -60 \\ -60 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -60 \\ -60 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -60 \\ -120 \end{bmatrix} = \begin{bmatrix} -20 \\ -40 \end{bmatrix}$$

$$\vec{x}(t) = \begin{bmatrix} 60 \\ 60 \end{bmatrix} - 20e^{-4t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} - 40e^{-t}$$

$x_1(t) = 60 - 20e^{-4t} - 40e^{-t}$

$x_2(t) = 60 + 20e^{-4t} - 80e^{-t}$

4) A focus in this course is a careful analysis of the mathematics and physical phenomena exhibited in forced and unforced mechanical (or electrical) oscillation problems. Using the mass-spring model, we've studied the differential equation for functions $x(t)$ solving

$$m x'' + c x' + k x = F_0 \cos(\omega t)$$

with $m, k, \omega > 0$; $c, F_0 \geq 0$.

When the forcing amplitude F_0 is positive there are two related but not identical phenomena known as "pure resonance" and "practical resonance" that can arise. Explain what each of these is, how they arise (in terms of the the parameters m, k, ω, c), how they are similar and how they are different.

(5 points)

pure resonance : $c = 0$, $\omega = \omega_0 = \sqrt{\frac{k}{m}}$

$$m x'' + k x = F_0 \cos \omega_0 t$$

get linearly growing solths

$$x = x_p + x_H$$

$$= \underbrace{t(A\cos\omega_0 t + B\sin\omega_0 t)}_{\text{amplitude grows linearly, unbounded solths.}} + c_1 \cos\omega_0 t + c_2 \sin\omega_0 t$$

practical resonance, $c > 0$ but small, $\omega \approx \omega_0$

$$m x'' + c x' + k x = F_0 \cos\omega t$$

$$x = x_p + x_H = x_{sp} + x_{tr}$$

$$= (A\cos\omega t + B\sin\omega t) + x_H$$

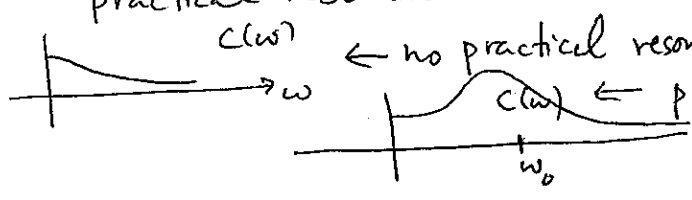
$$x = \underbrace{C(\cos(\omega t - \alpha))}_{C = C(\omega)} + x_H$$

$x_H \to 0$ as $t \to \infty$ because $c > 0$ $\Rightarrow$ x_H is exponentially decaying.

$C = C(\omega)$ can be relatively large for $\omega \approx \omega_0$

the damping prevents unbounded solutions but the oscillations can still be large if damping coefficient c is close to zero

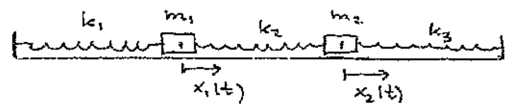
If amplitude $C = C(\omega)$ of steady periodic solth attains its maximum value for some $\omega > 0$, we have practical resonance



← no practical resonance

← practical resonance

5) Consider the following undamped mass-spring configuration, where as usual $x_1(t), x_2(t)$ measure the displacement of the two masses from equilibrium.



5a) In case $m_1 = 2$, $m_2 = 1$, $k_1 = 4$, $k_2 = 2$, $k_3 = 0$ (in other words, the third spring isn't really there), show that the resulting system of differential equations reduces to

$$x_1''(t) = -3x_1 + x_2$$
$$x_2''(t) = 2x_1 - 2x_2.$$

(5 points)

$m_1 x_1'' = -k_1 x_1 + k_2(x_2 - x_1)$

$m_1 x_1'' = -(k_1 + k_2)x_1 + k_2 x_2$

$2x_1'' = -(6)x_1 + 2x_2$

$\boxed{x_1'' = -3x_1 + x_2}$

$m_2 x_2'' = -k_2(x_2 - x_1) - k_3 x_2$

$m_2 x_2'' = k_2 x_1 - (k_2 + k_3)x_2$

$\boxed{x_2'' = 2x_1 - 2x_2}$

5b) What is the dimension of the solution space to this system of DEs? Explain.

(3 points)

4-dimensional.

is equivalent to a 1st order system of 4 (linear homog) DE's.

(also, 4 initial conds determine unique solution).

5c) Find the general solution to this system of differential equations. (Hint: The matrix in this second order system is the same as the matrix of the first order system in 3, and you may use that eigendata without rederiving it).

(8 points)

$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

from $\underline{3c}$

$\lambda = -4$ $\quad\quad \lambda = -1$

$\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\quad\quad \vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\omega = \sqrt{-\lambda} = 2$ $\quad\quad \omega = \sqrt{-\lambda} = 1$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = (c_1 \cos 2t + c_2 \sin 2t)\begin{bmatrix} 1 \\ -1 \end{bmatrix} + (c_3 \cos t + c_4 \sin t)\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

5d) Describe the two fundamental modes of this system.

(4 points)

slow in-phase mode, $\omega = 1$,
mass 2 has twice amplitude of mass 1

fast, out of phase mode, $\omega = 2$
mass 1 & mass 2 equal amplitude.