

Name.....

I.D. number.....

**Math 2280-001**  
**Second Midterm Exam**  
April 3, 2015

This exam is closed-book and closed-note. You may use a scientific calculator, but not one which is capable of graphing or of solving differential or linear algebra equations. **In order to receive full or partial credit on any problem, you must show all of your work and justify your conclusions.** There are 100 points possible, and the point values for each problem are indicated in the right-hand margin.  
**Good Luck!**

	Score	possible
1	_____	20
2	_____	25
3	_____	30
4	_____	5
5	_____	20
total	_____	100

1) Consider the following initial value problem, which could arise from Newton's second law in a forced mass-spring oscillation problem:

$$\begin{aligned}x''(t) + 2x'(t) + 10x(t) &= 26 \sin(2t) \\x(0) &= 0 \\x'(0) &= 6.\end{aligned}$$

1a) As a step in solving this IVP, find the general solution to the (underdamped) homogenous differential equation

$$x''(t) + 2x'(t) + 10x(t) = 0$$

(5 points)

1b) Find a particular solution to

$$x''(t) + 2x'(t) + 10x(t) = 26 \sin(2t)$$

using the method of undetermined coefficients.

(10 points)

1c) Use your work from parts a,b to solve the initial value problem at the top of this page.

(5 points)

2a) The homogeneous differential equation that you solved in 1a, namely

$$x''(t) + 2x'(t) + 10x(t) = 0$$

corresponds to the following first order system of two differential equations:

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -10 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Carefully explain the correspondence between solutions  $x(t)$  to the second order differential equation and

solutions  $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$  to the first order system.

(5 points)

2b) Use your work from 1a) to write down the general (homogenous) solution to the first order system

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -10 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Hint: It would be ugly to solve this problem using the eigenvalue-eigenvector methods of Chapter 5.

(5 points)

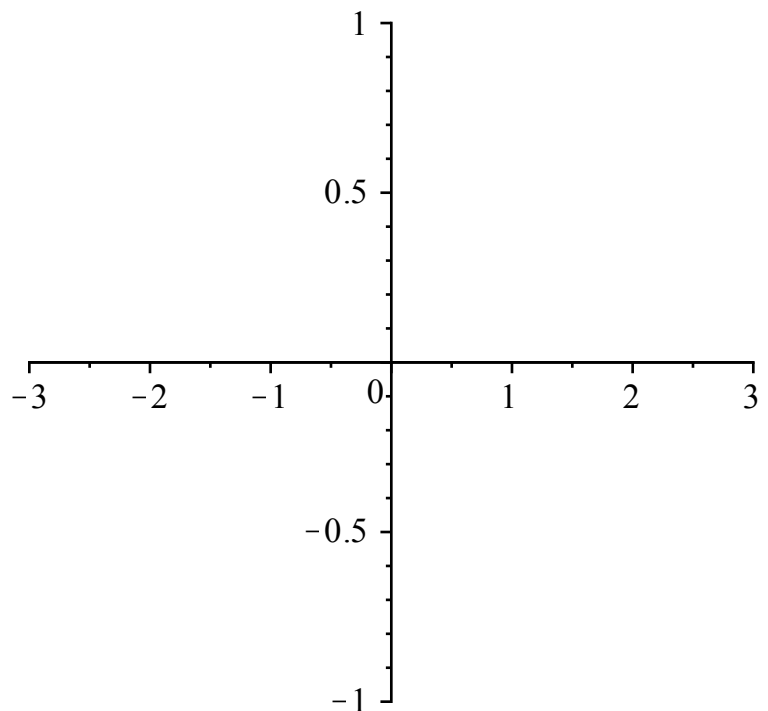
2c) Find the (complex) eigenvalues for the matrix in 2b. Use this information to classify the phase portrait as one of: nodal source, nodal sink, saddle, spiral source, spiral sink, center. For your convenience, the system is repeated here:

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -10 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

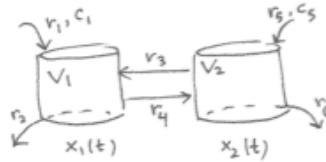
(5 points)

2d) Sketch the phase diagram for the system in 2b. Hint: It is possible to construct an adequate picture by using the classification in 2c and by checking the tangent vector field in several places.

(10 points)



3) Consider a general input-output model with two compartments as indicated below. The compartments contain volumes  $V_1, V_2$  and solute amounts  $x_1(t), x_2(t)$  respectively. The flow rates (volume per time) are indicated by  $r_i, i = 1 \dots 6$ . The two input concentrations (solute amount per volume) are  $c_1, c_5$ .



3a) Suppose  $r_2 = r_3 = r_6 = 100, r_1 = r_4 = 200, r_5 = 0 \frac{\text{gal}}{\text{hour}}$ . Explain why the volumes  $V_1(t), V_2(t)$  remain constant.

(4 points)

3b) Using the flow rates above,  $c_1 = 0.6, c_5 = 0 \frac{\text{lb}}{\text{gal}}, V_1 = V_2 = 100 \text{ gal}$ , show that the amounts of solute  $x_1(t)$  in tank 1 and  $x_2(t)$  in tank 2 satisfy

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 120 \\ 0 \end{bmatrix}.$$

(6 points)

3c) Find the general solution to the system

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 120 \\ 0 \end{bmatrix},$$

Hint: As steps you will want to find the homogeneous solution  $\underline{x}_{HP}$  and then a particular solution which is a constant vector,  $\underline{x}_p = \underline{c}$ .

(15 points)

3d) Solve the initial value problem for 3c, assuming there is initially no solute in either tank.

(5 points)

4) A focus in this course is a careful analysis of the mathematics and physical phenomena exhibited in forced and unforced mechanical (or electrical) oscillation problems. Using the mass-spring model, we've studied the differential equation for functions  $x(t)$  solving

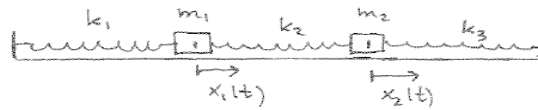
$$m x'' + c x' + kx = F_0 \cos(\omega t)$$

with  $m, k, \omega > 0$ ;  $c, F_0 \geq 0$ .

When the forcing amplitude  $F_0$  is positive there are two related but not identical phenomena known as "pure resonance" and "practical resonance" that can arise. Explain what each of these is, how they arise (in terms of the parameters  $m, k, \omega, c$ ), how they are similar and how they are different.

(5 points)

5) Consider the following undamped mass-spring configuration, where as usual  $x_1(t)$ ,  $x_2(t)$  measure the displacement of the two masses from equilibrium.



5a) In case  $m_1 = 2$ ,  $m_2 = 1$ ,  $k_1 = 4$ ,  $k_2 = 2$ ,  $k_3 = 0$  (in other words, the third spring isn't really there), show that the resulting system of differential equations reduces to

$$\begin{aligned} x_1''(t) &= -3x_1 + x_2 \\ x_2''(t) &= 2x_1 - 2x_2. \end{aligned}$$

(5 points)

5b) What is the dimension of the solution space to this system of DEs? Explain.

(3 points)

5c) Find the general solution to this system of differential equations. (Hint: The matrix in this second order system is the same as the matrix of the first order system in 3, and you may use that eigendata without rederiving it).

(8 points)

5d) Describe the two fundamental modes of this system.

(4 points)