Name ____

Student I.D. ___________________________________

Math 2280-001
Exam #1
February 20, 2015

Please show all work for full credit. This exam is closed book and closed note. You may use a scientific calculator, but not a "graphing calculator" i.e. not one which is capable of integration, taking derivatives, or matrix algebra. **In order to receive full or partial credit on any problem, you must show all of your work** and **justify your conclusions.** There are 100 points possible. The point values for each problem are indicated in the right-hand margin. **Good Luck!**

1		10
2		20
3		25
4		10
5		25
6		10
TOTAL		100

1) Concepts

1a) Define what it means for a first order differential equation for a function $y(x)$ to be separable.

(3 points)

A first order DE is separable provided it can be written in the form

$$y'(x) = f(x)g(y)$$

1b) Define what it means for a first order differential equation for a function $x(t)$ to be linear .

(3 points)

A first order DE for x(t) is linear provided it can be written in the form

$$x'(t) + p(t)x(t) = q(t)$$

1c) Define what it means for a first order differential equation for a function $x(t)$ to be autonomous

(2 points)

A first order DE is autonomous if it can be written as

$$x'(t) = f(x)$$

1d) Define what it means to be an equilibrium solution for a first order autonomous differential equation, for a function $x(t)$.

(2 points)

an equilibrium solution for an autonomous DE as in 1c is a constant solution $x(t) \equiv c$. (Thus c must be a root of f, $f(c) = 0$).

2) Consider the following input-output model: There is a tank containing 1000 liters of water, with an initial solute concentration of 0.01 $\frac{kg}{liter}$. At time $t = 0$, inflow and outflow pipes are activated, with water flowing into and out of the tank at a rate of 20 liters per minute. The water flowing into the tank contains solute in a concentration of 0.02 $\frac{kg}{l}$. Assume the water in the tank is well-mixed, so that at any instant the concentration everywhere in the tank is the essentially the same (although it varies in time).

2a) Use your modeling ability and the description above, to derive the following initial value problem for the mass $x(t)$ kg of solute at time t:

$$x'(t) = 0.4 - 0.02\,x$$
$$x(0) = 10$$

(5 points)

$$x'(t) = r_i c_i - r_o c_o.$$

From the description, $r_i = 20\ \frac{l}{\min}$, $c_i = .02\ \frac{kg}{l}$, $r_0 = 20\ \frac{l}{\min}$, $c_o = \frac{x}{V} = \frac{x}{1000}\ \frac{kg}{l}$. Thus

$$x'(t) = 20\cdot .02 - \frac{20\cdot x}{1000}$$
$$x'(t) = 0.4 - .02\,x.$$

Since the initial concentration in the tank is .01 $\frac{kg}{l}$ *the initial amount* $x(0)$ *is* $.01\cdot 1000 = 10$ *kg.*

2b) Solve the initial value problem in part a, using the algorithm for linear differential equations that we learned in Chapter 1

(10 points)

solution:

$$x'(t) + .02\,x(t) = 0.4$$
$$e^{.02\,t}(x' + .02\,x) = .4\,e^{.02\,t}$$
$$D_t\left(e^{.02\,t}x(t)\right) = .4\,e^{.02\,t}$$
$$e^{.02\,t}x(t) = \int .4\,e^{.02\,t}dt = \frac{.4}{.02}e^{.02\,t} + C = 20e^{.02\,t} + C$$
$$x(t) = 20 + Ce^{-.02\,t}.$$
$$x(0) = 10 \Rightarrow C = -10;$$
$$x(t) = 20 - 10e^{-.02\,t}$$

2c) Use your solution formula to deduce the limiting amount of salt in the tank, as $t \rightarrow \infty$. Explain why your answer makes sense in terms of the tank set-up and the in-flow solute concentration.

(5 points)

$$\lim_{t\rightarrow\infty} 20 - 10e^{-.02\,t} = 20\ kg.$$

This limit makes sense since we expect the limiting concentration to be the input concentration of $.02\ \frac{kg}{l}$*; since there are 1000 l in the tank this would imply that the limiting amount is* $.02\cdot 1000 = 20$ *kg.*

3) Consider the following differential equation for a population $P(t)$.

$$P'(t) = -2P^2 + 10P - 8.$$

3a) We considered differential equations of this nature in our discussions of applications. Describe what sort of population model and situation could lead to this differential equation.

(2 points)

This is a logistic population ($P'(t) = -2P^2 + 10P$), which is being harvested a a constant rate of 8 *(population units per unit of time).*

3b) Construct a phase diagram for this differential equation, and indicate the stability of the equilibrium solutions.

(6 points)

$$P'(t) = -2P^2 + 10P - 8 = -2\left(P^2 - 5P + 4\right)$$
$$P'(t) = -2(P-4)(P-1).$$

Thus the equilibrium solutions are $P \equiv 4$, $P \equiv 1$. *For* $P > 4$, $P'(t) < 0$; *For* $0 < P < 4$, $P'(t) > 0$; *For* $P < 0$, $P'(t) < 0$. *Thus the phase diagram is*

$$\leftarrow\leftarrow 1 \rightarrow\rightarrow 4 \leftarrow\leftarrow .$$

From the phase diagram, $P = 1$ *is an unstable equilibrium point and* $P = 4$ *is an asymptotically stable equilibrium point.*

3c) Use your favorite technique to find the partial fractions decomposition for

$$\frac{1}{(P-4)(P-1)}.$$

(7 points)

since

$$\frac{1}{P-4} - \frac{1}{P-1} = \frac{P-1-(P-4)}{(P-4)(P-1)} = \frac{3}{(P-4)(P-1)}$$

we see that

$$\frac{1}{(P-4)(P-1)} = \frac{1}{3}\left(\frac{1}{P-4} - \frac{1}{P-1}\right).$$

3d) Solve the IVP

$$P'(t) = -2P^2 + 10P - 8$$
$$P(0) = 2.$$

Your work from 3c should be helpful, and your work from 3b should let you check whether your solution seems reasonable.

(10 points)

$$\frac{dP}{dt} = -2(P-4)(P-1)$$
$$\frac{dP}{(P-4)(P-1)} = -2\,dt$$
$$\frac{1}{3}\left(\frac{1}{P-4} - \frac{1}{P-1}\right)dP = -2\,dt$$

$$\left(\frac{1}{P-4} - \frac{1}{P-1}\right)dP = -6\,dt$$
$$\int\left(\frac{1}{P-4} - \frac{1}{P-1}\right)dP = \int -6\,dt = -6t + C_1$$
$$\ln\left|\frac{P-4}{P-1}\right| = -6t + C_1$$
$$\left|\frac{P-4}{P-1}\right| = e^{-6t + C_1} = e^{C_1}e^{-6t}$$
$$\frac{P-4}{P-1} = Ce^{-6t}.$$

$P(0) = 2 \Rightarrow$

$$-\frac{2}{1} = C \Rightarrow \frac{P-4}{P-1} = -2e^{-6t}.$$
$$P - 4 = -2e^{-6t}(P-1)$$
$$P\left(1 + 2e^{-6t}\right) = 4 + 2e^{-6t}$$
$$P(t) = \frac{4 + 2e^{-6t}}{1 + 2e^{-6t}}.$$

Note that the initial value checks out, as does the phase portrait consequence that $\lim_{t \to \infty} P(t) = 4.$

4) Consider the initial value problem for a function $x(t)$:

$$x'(t) = 4t - 2x^3$$
$$x(0) = 1\,.$$

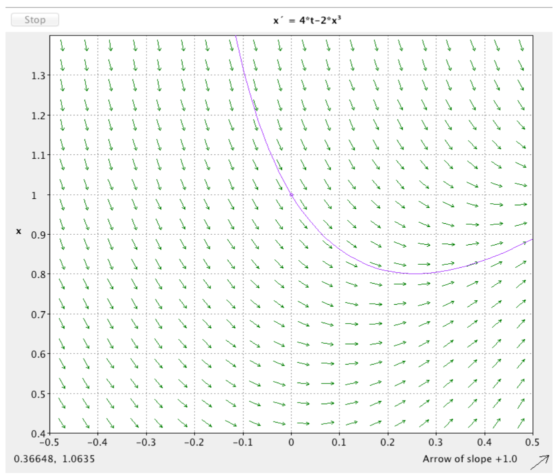
4a) Use the existence-uniqueness theorem to verify that this IVP has a unique solution (on some t interval).

(4 points)

The slope function $f(t, x) = 4t - 2x^3$ *is continuous in the* (t, x) *variable on all of* $\mathbb{R}^2$*, so solutions exist; since its partial derivative* $\frac{\partial}{\partial x} f(t, x) = -6x^2$ *is also continuous, solutions are unique.*

4b) Sketch the (approximate) graph of the solution to this initial value problem onto the slope field below.

(2 points)



4c) Use a single step of Euler's method, with step size $h = 0.1$, to estimate $x(0.1)$ for the solution to the initial value problem above. Add the corresponding approximate solution point to the slope field above. Do you expect the actual value for $x(0.1)$ to be larger or smaller than the Euler approximation?

(4 points)

The Euler method steps according to

$$t_{j+1} = t_j + h$$
$$x_{j+1} = x_j + h \cdot f(t_j, x_j)$$

At $(x_0, y_0) = (0, 1)$, $f(0, 1) = 0 - 2 = -2$. *Thus*

$$t_1 = .1$$
$$x_1 = 1 + .1 \cdot (-2) = 0.8$$

To complete this problem, one should plot the point $(0.1, 0.8)$ *on to the slope field above. Note that this will lie underneath the approximate graph of the solution, since the graph is concave up. (It appears that the exact value* $x(0.1) \approx 0.86$.

5) Consider a mass-spring configuration, with a mass $m = 2\ kg$ attached to a spring. The mass is set in motion so that at time t seconds its displacement $x(t)$ from equilibrium satisfies the differential equation

$$x''(t) + 2\,x'(t) + 10\,x(t) = 0.$$

5a) What are the values and units for the damping constant and Hooke's constant in this configuration? (Hint: note that the mass is $m = 2$ even though the coefficient of $x''(t)$ in the simplified differential equation above is 1.)

(5 points)

solution: the governing equation is

$$m\,x'' + c\,x' + k\,x = 0$$

so, because the mass is 2, *the original equation must have been twice the one that is displayed, i.e.*

$$2\,x''(t) + 4\,x'(t) + 20\,x(t) = 0.$$

Thus the Hooke's constant $k = 20\ \frac{N}{m}$ *and the damping constant is* $c = 4\ \frac{kg}{s}$. *(Note, the net units for each term in the governing equation must simplify to Newtons (* $= \frac{kg\cdot m}{s^2}$*), since the DE arises from Newton's second law.)*

5b) Find the general solution to the differential equation above.

(8 points)

trying $x(t) = e^{r\,t}$ yields $L(x) = e^{r\,t}p(r) = e^{r\,t}\left(r^2 + 2\,r + 10\right)$ *so exponential solutions must satisfy the characteristic equation*

$$r^2 + 2\,r + 10 = 0 \Rightarrow (r+1)^2 + 9 = 0$$

$$\Rightarrow (r+1)^2 = -9 \Rightarrow r + 1 = \pm\,3\,i \Rightarrow r = -1 \pm 3\,i.$$

Thus, the general homogeneous solution is

$$x(t) = c_1 e^{-t}\cos(3\,t) + c_2 e^{-t}\sin(3\,t)$$

5c) What sort of damping is exhibited by solutions to this differential equation?

(2 points)

underdamped. (infinitely oscillating, exponentially decaying solutions).

5d) Now consider the nonhomogeneous differential equation

$$x''(t) + 2x'(t) + 10x(t) = 5$$

(which could arise if a constant external force was applied to the mass in the original configuration). Notice that this differential equation has a constant solution $x(t) \equiv 0.5$. Solve the initial value problem for this nonhomogeneous DE, with initial conditions

$$x(0) = 1.5$$
$$x'(0) = -1.$$

(10 points)

Using $x = x_P + x_H$, with $x_p = 0.5$ and x_H from c,

$$x(t) = 0.5 + c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t)$$

$$\Rightarrow x'(t) = c_1\left(-e^{-t}\cos(3t) - 3e^{-t}\sin(3t)\right) + c_2\left(-e^{-t}\sin(3t) + 3e^{-t}\cos(3t)\right).$$

Thus

$$x(0) = 1.5 = 0.5 + c_1$$
$$x'(0) = -1 = -c_1 + 3c_2.$$

The first equation implies $c_1 = 1$, and then the second equation implies $c_2 = 0$.

$$x(t) = 0.5 + e^{-t}\cos(3t).$$

6a) Let V, W be vector spaces. Define what it means for $L : V \rightarrow W$ to be a linear transformation.

(4 points)

L is linear means that it is always true (i.e. $\forall\ y_1, y_2 \in V, c \in \mathbb{R}$) that

$$L(y_1 + y_2) = L(y_1) + L(y_2)$$
$$L(c\,y) = c\,L(y).$$

6b) Prove that if $L : V \rightarrow W$ is a linear transformation, and if $y_P \in V$ solves the nonhomogeneous equation

$$L(y_P) = f$$

then every solution of the equation

$$L(y) = f$$

is of the form $y = y_P + y_H$, where y_H is some solution of the homogeneous equation

$$L(y) = 0.$$

(6 points)

Let $L(y_p) = f$. *Let* $L(y_H) = 0$. *Then*

$$L(y_p + y_H) = L(y_p) + L(y_H) = f + 0 = f.$$

This verifies that $y_p + y_H$ *also solves the equation* $L(y) = f$. *Conversely, let* y_q be any other solution, i.e. $L(y_q) = f$. *Then*

$$y_q = y_p + (y_q - y_p).$$

Compute $L(y_q - y_p) = L(y_q) + L(-y_p) = L(y_q) - L(y_p) = f - f = 0$. *Thus* y_q *is the sum of* y_p *with a homogeneous solution* $y_H = y_p - y_q$.