

Name _____

Student I.D. _____

Math 2280-001
Exam #1
February 20, 2014

Please show all work for full credit. This exam is closed book and closed note. You may use a scientific calculator, but not a "graphing calculator" i.e. not one which is capable of integration, taking derivatives, or matrix algebra. **In order to receive full or partial credit on any problem, you must show all of your work and justify your conclusions.** There are 100 points possible. The point values for each problem are indicated in the right-hand margin. **Good Luck!**

1	_____	10
2	_____	20
3	_____	25
4	_____	10
5	_____	25
6	_____	10
TOTAL	_____	100

1) Concepts

1a) Define what it means for a first order differential equation for a function $y(x)$ to be separable. (3 points)

1b) Define what it means for a first order differential equation for a function $x(t)$ to be linear. (3 points)

1c) Define what it means for a first order differential equation for a function $x(t)$ to be autonomous. (2 points)

1d) Define what it means to be an equilibrium solution for a first order autonomous differential equation, for a function $x(t)$. (2 points)

2) Consider the following input-output model: There is a tank containing 1000 liters of water, with an initial solute concentration of $0.01 \frac{\text{kg}}{\text{liter}}$. At time $t = 0$, inflow and outflow pipes are activated, with water flowing into and out of the tank at a rate of 20 liters per minute. The water flowing into the tank contains solute in a concentration of $0.02 \frac{\text{kg}}{\text{l}}$. Assume the water in the tank is well-mixed, so that at any instant the concentration everywhere in the tank is the essentially the same (although it varies in time).

2a) Use your modeling ability and the description above, to derive the following initial value problem for the mass $x(t)$ kg of solute at time t :

$$\begin{aligned}x'(t) &= 0.4 - 0.02x \\x(0) &= 10\end{aligned}$$

(5 points)

2b) Solve the initial value problem in part a, using the algorithm for linear differential equations that we learned in Chapter 1

(10 points)

2c) Use your solution formula to deduce the limiting amount of salt in the tank, as $t \rightarrow \infty$. Explain why your answer makes sense in terms of the tank set-up and the in-flow solute concentration.

(5 points)

3) Consider the following differential equation for a population $P(t)$.

$$P'(t) = -2P^2 + 10P - 8.$$

3a) We considered differential equations of this nature in our discussions of applications. Describe what sort of population model and situation could lead to this differential equation. (2 points)

3b) Construct a phase diagram for this differential equation, and indicate the stability of the equilibrium solutions. (6 points)

3c) Use your favorite technique to find the partial fractions decomposition for

$$\frac{1}{(P-4)(P-1)}.$$

(7 points)

3d) Solve the IVP

$$P'(t) = -2P^2 + 10P - 8$$
$$P(0) = 2.$$

Your work from 3c should be helpful, and your work from 3b should let you check whether your solution seems reasonable.

(10 points)

4) Consider the initial value problem for a function $x(t)$:

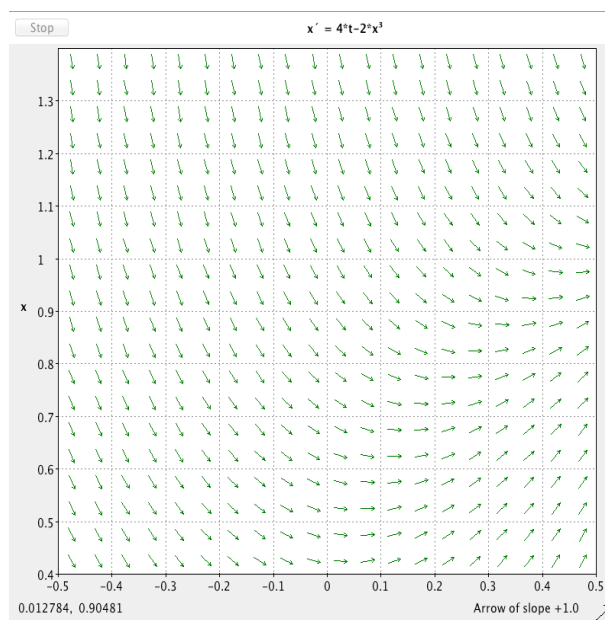
$$x'(t) = 4t - 2x^3$$

$$x(0) = 1.$$

4a) Use the existence-uniqueness theorem to verify that this IVP has a unique solution (on some t interval).

(4 points)

4b) Sketch the (approximate) graph of the solution to this initial value problem onto the slope field below. (2 points)



4c) Use a single step of Euler's method, with step size $h = 0.1$, to estimate $x(0.1)$ for the solution to the initial value problem above. Add the corresponding approximate solution point to the slope field above. Do you expect the actual value for $x(0.1)$ to be larger or smaller than the Euler approximation?

(4 points)

5) Consider a mass-spring configuration, with a mass $m = 2 \text{ kg}$ attached to a spring. The mass is set in motion so that at time t seconds its displacement $x(t)$ from equilibrium satisfies the differential equation

$$x''(t) + 2x'(t) + 10x(t) = 0.$$

5a) What are the values and units for the damping constant and Hooke's constant in this configuration? (Hint: note that the mass is $m = 2$ even though the coefficient of $x''(t)$ in the simplified differential equation above is 1.)

(5 points)

5b) Find the general solution to the differential equation above.

(8 points)

5c) What sort of damping is exhibited by solutions to this differential equation?

(2 points)

5d) Now consider the nonhomogeneous differential equation

$$x''(t) + 2x'(t) + 10x(t) = 5$$

(which could arise if a constant external force was applied to the mass in the original configuration).

Notice that this differential equation has a constant solution $x(t) \equiv 0.5$. Solve the initial value problem for this nonhomogeneous DE, with initial conditions

$$x(0) = 1.5$$

$$x'(0) = -1.$$

(10 points)

6a) Let V, W be vector spaces. Define what it means for $L : V \rightarrow W$ to be a linear transformation. (4 points)

6b) Prove that if $L : V \rightarrow W$ is a linear transformation, and if $y_P \in V$ solves the nonhomogeneous equation

$$L(y_P) = f$$

then every solution of the equation

$$L(y) = f$$

is of the form $y = y_P + y_H$ where y_H is some solution of the homogeneous equation

$$L(y) = 0.$$

(6 points)