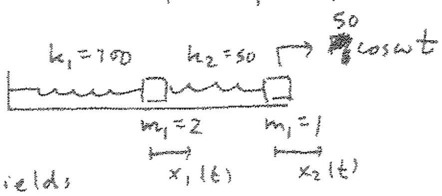
Wed Mar. 9

Math 2280-1

- unforced spring problems, pages 2-3 Tuesday §5.3
- Now force our system @ angular frequency ω, pulling/pushing on the second mass.

This is example 3 p. 327



yields

$$\vec{x}'' = \begin{bmatrix} -75 & 25 \\ 50 & -50 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 50 \end{bmatrix} \cos \omega t$$

$$\vec{x}'' = A\vec{x} + \vec{F}_0 \cos \omega t$$

$$A = M^{-1}K$$
$$\vec{F}_0 = M^{-1}\vec{C}$$

↑ original force amplitude vector

try $\vec{x}_p = \vec{c} \cos \omega t$ unknown $\vec{c}$

yields

$$-\omega^2 \vec{c} \, \cancel{\cos \omega t} = A\vec{c} \, \cancel{\cos \omega t} + \begin{bmatrix} 0 \\ 50 \end{bmatrix} \cancel{\cos \omega t}$$

$$-\omega^2 I \vec{c} = A\vec{c} + \begin{bmatrix} 0 \\ 50 \end{bmatrix}$$

$$\left[A + I\omega^2\right] \vec{c} = \begin{bmatrix} 0 \\ -50 \end{bmatrix}$$

$$\vec{c} = \left[A + I\omega^2\right]^{-1} \begin{bmatrix} 0 \\ -50 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -75+\omega^2 & 25 \\ 50 & -50+\omega^2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -50 \end{bmatrix}$$

$$= \frac{1}{(\omega^2-75)(\omega^2-50) - 50 \cdot 25} \begin{bmatrix} -50+\omega^2 & -25 \\ -50 & -75+\omega^2 \end{bmatrix} \begin{bmatrix} 0 \\ -50 \end{bmatrix}$$

$$\omega^4 - 125\omega^2 - 2500$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{(\omega^2-25)(\omega^2-100)} \begin{bmatrix} 1250 \\ 50(75-\omega^2) \end{bmatrix}$$

$$\vec{x}_p = \cos \omega t \, \vec{c}$$

$$-\omega^2 \cancel{\cos \omega t} \, \vec{c} = \cancel{\cos \omega t} \, A\vec{c} + \vec{F}_0 \cancel{\cos \omega t}$$

$$-\omega^2 I \vec{c} = A\vec{c} + \vec{F}_0$$

$$-\vec{F}_0 = (A + \omega^2 I)\vec{c}$$

$$\boxed{\vec{c} = -(A + \omega^2 I)^{-1} \vec{F}_0}$$

works unless $A + \omega^2 I$ is singular, i.e. unless $\omega^2 = -\lambda$ for some eigenvalue of A, i.e. unless we force at a natural frequency and induce resonance!

$\vec{x}_p = \cos \omega t \, \vec{c}$

notice that as ω approaches 5 or 10 $\vec{x}_p$ amplitudes $\to \infty$!

In a slightly damped system (where $\vec{x}_H \to 0$), this would mean practical resonance - see Maple!

MATH 2280-1
Mass–Spring systems
March 9, 2011

This handout covers section 5.3 and might help you for the Earthquake exploration you're doin gat the end of that section, on pages 330–332. You're mostly on your own (possibly with a partner) for this project, but here is a small example of a spring system worked out on Maple, so that you can get an idea about useful commands to use. We're working this example out by hand in class as well.

```
> with(LinearAlgebra): with(plots): with(DEtools):
> M:=Matrix([[2,0],[0,1]]);
  K:=Matrix([[-150,50],[50,-50]]);
  A:=M^(-1).K  ;#period to multiply
```

$$M := \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$K := \begin{bmatrix} -150 & 50 \\ 50 & -50 \end{bmatrix}$$

$$A := \begin{bmatrix} -75 & 25 \\ 50 & -50 \end{bmatrix} \qquad (1)$$

```
> Eigenvectors(A);
```

$$\begin{bmatrix} -100 \\ -25 \end{bmatrix}, \begin{bmatrix} -1 & \frac{1}{2} \\ 1 & 1 \end{bmatrix} \qquad (2)$$

Therefore, the natural frequencies of this system are the 10 and 5, and the two fundamental modes correspond to the masses moving in opposite directions (with equal amplitudes and angular frequency 10) and in parallel directions (with amplitude ratio of two and angular frequency 5), as we work out by hand today, on page 3 of Tuesday's notes (or elsewhere).

Now, let's consider the forced system with force vector equal to $\cos(\omega \cdot t)\,[50, 0]$, i.e. the second mass is being forced periodically. In other words, the system **Mx''=Kx + F**, where F=cos(wt)[0,50] ; this is Example 3 on page 327, and worked out in today's notes.

```
> F0 := M^(-1).Vector([0, 50]) :
        #The F0 in the normalized equation (30), page 326...
         # needs serious modification for Maple project
  Iden := Matrix(2, shape = identity) :
        # the 2 by 2 identity matrix
  Aleft := omega -> A + ω^2·Iden :
      # the matrix function multiplying
      # c on the left side of (32)
  c := omega -> (Aleft(omega))^(-1).F0 :
      # the solution vector c(omega) to (32),
      # obtained by multiplying both sides of equation
      # (32) on the left, by the inverse to Aleft

> c(omega);
```

$$\begin{bmatrix} -\dfrac{1250}{2500 - 125\,\omega^2 + \omega^4} \\[2ex] \dfrac{50\left(-75 + \omega^2\right)}{2500 - 125\,\omega^2 + \omega^4} \end{bmatrix} \tag{3}$$

The vector $c(\omega)$, as above, times the function of time, $\cos(\omega \cdot t)$, is a particular solution to the forced oscillation problem we are considering. If we assume that our actual problem has a small amount of damping, then we expect that this particular solution is very close to the steady periodic solution to the damped problem. Read the discussion on page 327. We can study practical resonaonace phenomena for these slightly damped problmes by plotting the maximum amplitude for the individual mass displacements, for these particular steady state solutions to the undamped problems. Use the Maple command "norm" to measure this maximum amplitude.

```
> norm(c(omega));
```

$$\max\left(\frac{1250}{\left|2500 - 125\,\omega^2 + \omega^4\right|}, 50\left|\frac{-75 + \omega^2}{2500 - 125\,\omega^2 + \omega^4}\right|\right) \quad (4)$$

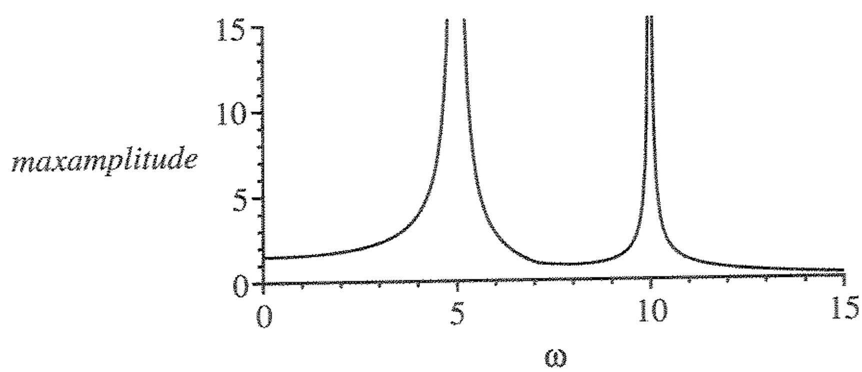
Another way to measure the size of $c(\omega)$ is to take the Euclidean magnitude, which is the command

```
> norm(c(omega), 2);
```

$$50\sqrt{\frac{625}{\left|2500 - 125\,\omega^2 + \omega^4\right|^2} + \left|\frac{-75 + \omega^2}{2500 - 125\,\omega^2 + \omega^4}\right|^2} \quad (5)$$

(You will use the first command in your Maple project, which perhaps makes the most sense since it will be measuring the maximum amplitude that any floor oscillates relative to the ground.) The following picture illustrates that the maximum amplitutde of the particular solution blows up when ω is near the two natural angular frequencies. Thus, in the slightly damped problem, one would experience practical resonance in the steady periodic solution (which would be very close to the steady periodic solution in our undamped problem).

```
> plot(norm(c(omega)),omega=0..15,maxamplitude=0..15,
        numpoints=200,color='black');
```



This is qualitatively the picture on page 327, figure 5.3.10, although they plotted the (Euclidean) magnitude of c(omega) rather than the maximum of the individual amplitudes. Notice how we get Maple to label the axes as desired.

We can get a plot of resonance as a function of period by recalling that $\frac{2\cdot\pi}{T}=\omega$:

```
> plot(norm(c(2*Pi/period)),period=0.1..3,maxamplitude=0..15,
  numpoints=200,color='black');
```