Math 2280-1
Tues. 3/29

Wed: introduce Chapter 6
Thurs: go over last year's exam

finish variation of parameters discussion,
in particular derive the formulas on page 5 of Monday's notes to directly solve IVP's, for non-homogeneous systems

$$\begin{cases} \vec{x}\,' - P(t)\vec{x} = \vec{f}(t) \\ \vec{x}(0) = \vec{x}_0 \end{cases}$$

Marvel how in some sense 1^{st} order systems of linear DE's are no harder than scalar 1^{st} order linear DE's in §1.5

• example on pages 6-7 of Monday's notes had a Maple typo, which I've fixed below.

$$\begin{cases} \vec{x}\,' = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \vec{x} - \begin{bmatrix} 15 \\ 4 \end{bmatrix} t e^{-2t} \\ \vec{x}(0) = \begin{bmatrix} 7 \\ 3 \end{bmatrix} \end{cases}$$

work out

$$\boxed{\vec{x}(t) = e^{At}\vec{x}_0 + e^{At}\int_0^t e^{-As}\vec{f}(s)\,ds}$$

check some of these steps by hand:

Maple work for section 5.6 variation of parameters example:

```
> with(LinearAlgebra) :
  A := Matrix(2, 2, [4, 2, 3, -1]);
```

$$A := \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$$

```
> Eigenvectors(A);
```

$$\begin{bmatrix} 5 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 & -\frac{1}{3} \\ 1 & 1 \end{bmatrix}$$

```
> Phi := t→(exp(5·t)· Vector([2, 1])|exp(-2·t)·Vector([-1, 3])) :
  # FMS by augmenting two columns
> Phi(t);
```

$$\begin{bmatrix} 2e^{5t} & -e^{-2t} \\ e^{5t} & 3e^{-2t} \end{bmatrix}$$

```
> Phi(t).Phi(0)^(-1);
  MatrixExponential(A, t); # should agree with first command
```

$$\begin{bmatrix} \frac{6}{7}e^{5t}+\frac{1}{7}e^{-2t} & \frac{2}{7}e^{5t}-\frac{2}{7}e^{-2t} \\ \frac{3}{7}e^{5t}-\frac{3}{7}e^{-2t} & \frac{1}{7}e^{5t}+\frac{6}{7}e^{-2t} \end{bmatrix}$$

$$\begin{bmatrix} \frac{6}{7}e^{5t}+\frac{1}{7}e^{-2t} & \frac{2}{7}e^{5t}-\frac{2}{7}e^{-2t} \\ \frac{3}{7}e^{5t}-\frac{3}{7}e^{-2t} & \frac{1}{7}e^{5t}+\frac{6}{7}e^{-2t} \end{bmatrix}$$

```
> x0 := Vector([7, 3]) :
  f := t -> -t·exp(-2·t)·Vector([15, 4]) :
     #Monday's notes were missing a minus sign for f
  f(s);
```

$$\begin{bmatrix} -15\,s\,e^{-2s} \\ -4\,s\,e^{-2s} \end{bmatrix}$$

```
> g := s -> simplify(MatrixExponential(A, -s).f(s)) :
  g(s);
```

$$\begin{bmatrix} -s\,(14\,e^{-7s}+1) \\ -s\,(-3+7\,e^{-7s}) \end{bmatrix}$$

```
> G := t -> Vector([int(g(s)[1], s=0..t), int(g(s)[2], s=0..t)]);
```

$$G := t \rightarrow \mathit{Vector}\left(\left[\int_0^t g(s)_1\,ds, \int_0^t g(s)_2\,ds\right]\right)$$

```
> G(t);
```

$$\begin{bmatrix} -\frac{2}{7}+2\,t\,e^{-7t}+\frac{2}{7}e^{-7t}-\frac{1}{2}t^2 \\ -\frac{1}{7}+\frac{3}{2}t^2+t\,e^{-7t}+\frac{1}{7}e^{-7t} \end{bmatrix}$$

```
> x := t -> simplify(MatrixExponential(A, t).(x0 + G(t))) :
     #variation of parameters solution formula
> x(t); #should agree with page 368 formula
```

$$\begin{bmatrix} \frac{46}{7}e^{5t}+2\,e^{-2t}\,t+\frac{3}{7}e^{-2t}-\frac{1}{2}e^{-2t}t^2 \\ \frac{23}{7}e^{5t}+e^{-2t}\,t-\frac{2}{7}e^{-2t}+\frac{3}{2}e^{-2t}t^2 \end{bmatrix}$$

```
>
```

how does this compare
to undetermined coeff's

$\vec{x}(t) = \vec{x}_p(t) + \vec{x}_H(t)$?

Odds & Ends from Chapter 5: page 2 of March 18 notes:

2

Chain-free way to get $\Phi(t)$: $|A-\lambda I|$ has factor $(\lambda-\lambda_j)^{k_j}$ with k_j = alg mult

Then $G_{\lambda_j} = \ker(A-\lambda_j)^{k_j}$ has dim k_j

pick a basis $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_{k_j}$ for this generalized eigenspace. (In practice, Maple might help.)

Then $\vec{x}_\ell(t) = e^{At}\vec{v}_\ell$ is the soltn to $\begin{cases} \vec{x}'(t) = A\vec{x} \\ \vec{x}(0) = \vec{v}_\ell \end{cases}$

$$= e^{\lambda_j I t} e^{(A-\lambda_j I)t}\vec{v}_\ell$$

$$= e^{\lambda_j t}\left[e^{(A-\lambda_j I)t}\vec{v}_\ell\right]$$

$$= e^{\lambda_j t}\left[I + (A-\lambda_j I)t + \ldots\right]\vec{v}_\ell$$

$$= e^{\lambda_j t}\left[\vec{v}_\ell + t(A-\lambda_j I)\vec{v}_\ell + \frac{t^2}{2}(A-\lambda_j I)^2\vec{v}_\ell + \ldots \frac{t^{k-1}}{(k-1)!}(A-\lambda_j I)^{k-1}\vec{v}_\ell + \vec{O}\right]$$

finite sum! $\vec{v}_\ell \in \ker(A-\lambda_j I)^k$!

this gives k_j soltns "for" λ_j
amalgamate to get $\Phi(t)$!

example con't $A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 3 & -1 \\ 0 & 0 & 3 \end{bmatrix}$ (still) G_3 has dim 3, i.e. is $\mathbb{R}^3$
so, pick $\vec{v}_1 = \vec{e}_1$, $\vec{v}_2 = \vec{e}_2$, $\vec{v}_3 = \vec{e}_3$.

$$\vec{x}_1(t) = e^{At}\vec{e}_1 = e^{3t}e^{(A-3I)t}\vec{e}_1 = e^{3t}\Big[I\vec{e}_1 + \underbrace{(A-3I)t\vec{e}_1}_{\text{already zero}} + \ldots\Big] = e^{3t}\begin{bmatrix}1\\0\\0\end{bmatrix}.$$

$$\vec{x}_2(t) = e^{At}\vec{e}_2 = e^{3t}e^{(A-3I)t}\vec{e}_2 = e^{3t}\Big[I\vec{e}_2 + \begin{bmatrix}0&1&2\\0&0&-1\\0&0&0\end{bmatrix}t\vec{e}_2 + \underbrace{\begin{bmatrix}0&0&-1\\0&0&0\\0&0&0\end{bmatrix}\frac{t^2}{2}\vec{e}_2}_{\text{already zero}} + \ldots\Big]$$

$$= e^{3t}\begin{bmatrix}t\\1\\0\end{bmatrix}$$

$$\vec{x}_3(t) = e^{At}\vec{e}_3 = e^{3t}e^{(A-3I)t}\vec{e}_3$$

$$= e^{3t}\Big[I\vec{e}_3 + \underbrace{\begin{bmatrix}0&1&2\\0&0&-1\\0&0&0\end{bmatrix}t\vec{e}_3}_{\begin{bmatrix}2t\\-t\\0\end{bmatrix}} + \underbrace{\begin{bmatrix}0&0&-1\\0&0&0\\0&0&0\end{bmatrix}\frac{t^2}{2}\vec{e}_3}_{\begin{bmatrix}-t^2/2\\0\\0\end{bmatrix}} + \vec{0}\Big]$$

$$= e^{3t}\begin{bmatrix}2t - t^2/2\\-t\\1\end{bmatrix}$$

$$\Phi(t) = \left[\vec{x}_1 \middle| \vec{x}_2 \middle| \vec{x}_3\right] = e^{3t}\begin{bmatrix}1 & t & 2t-t^2/2\\0&1&-t\\0&0&1\end{bmatrix}$$; also equals e^{At}, in this case

Exam material review

to cover 3.6–3.7, 4.1, 5.1–5.6

Chapter 3 3.6–3.7

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- forced oscillations

  $m x'' + c x' + k x = F_0 \cos\omega t$

  $c = 0$ non-resonance, beating, resonance

  $c > 0$ $x_{sp}(t)$, $x_{tr}(t)$, practical resonance

- using KE + PE = const in conservative systems to deduce natural frequencies

skip 3.7 (electrical circuits)

Chapter 4 4.1 theory for 1^{st} order systems of linear DE's.

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- $\exists!$ thm statement for 1^{st} order linear IVP
- deduction of dim of soltn space for homogeneous 1^{st} order linear sys. DE's.
- converting DE's (or systems) into equivalent 1^{st} order systems (and deducing dim of soltn space for homogeneous linear ones)

skip 4.3 (numerical methods)

Chapter 5 linear systems of DE's

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- eval-evect method for $\vec{x}' = A\vec{x}$
  - diagonalizable case
  - complex evals/evects
  - defective eigenspaces.
  - application to input/output modeling (tanks).
- $-\omega^2 = \lambda$ method for $\vec{x}'' = A\vec{x}$ if $A = M^{-1}K$ from a conservative spring-mass system
  - undamped spring systems
  - $\vec{x}_p = \vec{c}\cos\omega t$, for $\vec{x}'' - A\vec{x} = \vec{b}\cos\omega t$ (to understand practical resonance phenomena with damping $\ll 1$).
- FMS $\Phi(t)$ for $\vec{x}' = A\vec{x}$
  - e^{At}
    - power series
    - $\Phi(t)\,\Phi(0)^{-1}$
- $\vec{x}' - A\vec{x} = \vec{f}(t)$
  - undetermined coefficients for $\vec{x}_p$
  - variation of parameters
    - special case where $\Phi(t) = e^{At}$, esp. for IVP $\begin{cases} \vec{x}' - A\vec{x} = \vec{f} \\ \vec{x}(0) = \vec{x}_0. \end{cases}$
  - applications to input/output models