

Math 2280-1
Monday March 29

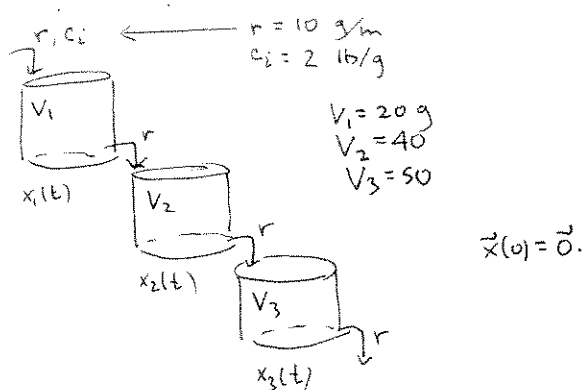
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§ 5.6 Non homogeneous linear systems of DE's
(will use e^{At} & FMS's)

$$\frac{d\vec{x}}{dt} = P(t)\vec{x} + \vec{f}(t)$$

- undetermined coeff's in case $P(t) = A$ (const matrix)
and $\vec{f}(t)$ is one of good guessing functions
- variation of parameters ($\dot{\cdot}$), for general $P(t)$, $\vec{f}(t)$, provided
you already have a FMS $\vec{\Phi}(t)$ for the homogeneous
system $\frac{d\vec{x}}{dt} = P(t)\vec{x}$.

Example p. 363 (Example 2)



Explain why the 1st order system for $\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$ is

$$* \quad \frac{d\vec{x}}{dt} = \begin{bmatrix} -.5 & 0 & 0 \\ .5 & -.25 & 0 \\ 0 & .25 & -.2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 20 \\ 0 \\ 0 \end{bmatrix}$$

general sol'n is

$$\vec{x}(t) = \vec{x}_H(t) + \vec{x}_P(t)$$

as usual.

(2)

$$\text{for } A = \begin{bmatrix} -.5 & 0 & 0 \\ .5 & -.25 & 0 \\ 0 & .25 & -.2 \end{bmatrix}$$

Maple says

$$\lambda = -.5$$

$$\lambda = -.25$$

$$\lambda = -.2$$

$$\vec{v} = \begin{bmatrix} 5 \\ -6 \\ 5 \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(plus, we analyzed this matrix before!)

so, for $\vec{x}_H(t)$ a FMS $\Phi(t)$ is

$$\Phi(t) = \begin{bmatrix} \\ \\ \end{bmatrix}$$

and general sol'n to $\vec{x}_H(t) = \Phi(t) \vec{c}$.For $\vec{x}_p(t)$ we guess a constant vector, since

$$L(\vec{x}) := \vec{x}' - A\vec{x}$$

transforms constant vectors to constant vectors [This is undetermined coeff's easy case for systems!]

If $\vec{x}_p(t) = \vec{k}$, then plug into DE: $\vec{x}' = A\vec{x} + \vec{b}$, page 1

$$\vec{0} = \begin{bmatrix} -.5 & 0 & 0 \\ .5 & -.25 & 0 \\ 0 & .25 & -.2 \end{bmatrix} \vec{k} + \begin{bmatrix} 20 \\ 0 \\ 0 \end{bmatrix};$$

$$\vec{k} = A^{-1} \begin{bmatrix} -20 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 40 \\ 80 \\ 100 \end{bmatrix} \quad (\text{Maple})$$

general sol'n

$$\vec{x}(t) = \vec{k} + \Phi(t) \vec{c} \quad (\vec{x}_p + \vec{x}_H), \quad \text{with } \Phi(t) = \left[e^{-.5t} \begin{bmatrix} 5 \\ -6 \\ 5 \end{bmatrix}, e^{-.25t} \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix}, e^{-.2t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right]$$

If $\vec{x}(0) = \vec{0}$ (as in example),

$$\vec{0} = \vec{k} + \Phi(0) \vec{c}$$

$$\vec{c} = \Phi(0)^{-1} (-\vec{k}) = \begin{bmatrix} -40/3 \\ -160 \\ -2500/3 \end{bmatrix} \quad (\text{Maple})$$

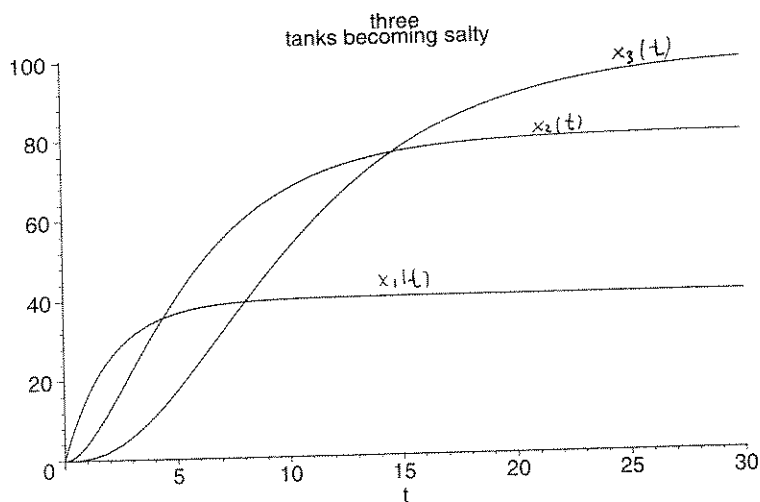
(3)

so $\vec{x}(t) = \begin{bmatrix} 40 \\ 80 \\ 100 \end{bmatrix} + \frac{-40}{3} e^{-.5t} \begin{bmatrix} 3 \\ -6 \\ 5 \end{bmatrix} - 160 e^{-.25t} \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix} - \frac{2500}{3} e^{-.2t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

```
> x1:=t->40-40*exp(-.5*t);
  x2:=t->80+80*exp(-.5*t)-160*exp(-.25*t);
  x3:=t->100-200/3*exp(-.5*t)+5*160*exp(-.25*t)-
    2500/3*exp(-.2*t);

  x1:=t->40-40*e(-0.5 t)
  x2:=t->80+80*e(-0.5 t)-160*e(-0.25 t)
  x3:=t->100-200/3*e(-0.5 t)+800*e(-0.25 t)-2500/3*e(-0.2 t)

> plot({x1(t),x2(t),x3(t)},t=0..30,color=black,title='three
tanks becoming salty');
```



but, for harder nonhomogeneous problems, you'll prefer to use variation of parameters! ☺

Variation of parameters, revisited !!

Consider the non-homogeneous linear system of DE's

$$* \quad \vec{x}'(t) = P(t)\vec{x} + \vec{f}(t)$$

Let $\Phi(t)$ be a FMS for the homogeneous system

$$\vec{x}'(t) = P(t)\vec{x}$$

Since $\Phi(t)$ is invertible $\forall t \in I$,

any solution $\vec{x}(t)$ to $*$ can be written as

$$\vec{x}(t) = \Phi(t)\vec{u}(t) \quad (\text{i.e. } \vec{u} = \Phi^{-1}\vec{x})$$

plug into $*$:

$$\cancel{\Phi'}\vec{u} + \Phi\vec{u}' = \cancel{P\Phi}\vec{u} + \vec{f}$$

$$\text{"P}\Phi \quad \text{i.e. } \Phi\vec{u}' = \vec{f}, \quad \text{i.e. } \boxed{\vec{u}' = \Phi^{-1}\vec{f}} \quad ; \quad \boxed{\vec{u} = \int \Phi^{-1}\vec{f} dt}$$

Exercise 1 (See p. 368 for case $n=2$)

When we were doing higher order linear eqns, we considered the DE (Feb 15, although we restricted to $n=2$ in class)

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y = f$$

For a basis $\{y_1, \dots, y_n\}$ of homogeneous sol's the var. par. formula for y_p was

$$y_p = u_1 y_1 + u_2 y_2 + \dots + u_n y_n$$

$$\text{where } \begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{bmatrix} = [W]^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f \end{bmatrix}$$

explain why this formula (which seemed specially mysterious at the time) actually a special case of the boxed formula above!

If you want to solve the IVP

$$\begin{cases} \vec{x}'(t) = P(t)\vec{x} + \vec{f}(t) \\ \vec{x}(0) = \vec{x}_0 \end{cases}$$

Then $\vec{x} = \Phi \vec{u}$ for $\vec{u}' = \Phi^{-1} \vec{f}$

$$\vec{x}_0 = \Phi(0) \vec{u}_0$$

$$\text{so } \vec{u}_0 = \Phi(0)^{-1} \vec{x}_0$$

$$\text{Thus } \vec{u}(t) = \vec{u}_0 + \int_0^t \vec{u}'(s) ds$$

$$\vec{u}(t) = \vec{u}_0 + \int_0^t \Phi^{-1}(s) \vec{f}(s) ds$$

$$\vec{x}(t) = \Phi(t) \vec{u}(t)$$

$$\vec{x}(t) = \Phi(t) \left[\Phi(0)^{-1} \vec{x}_0 + \int_0^t \Phi^{-1}(s) \vec{f}(s) ds \right]$$

If you want to solve the special case IVP ($P(t) = A, \text{const.}$)

$$\begin{cases} \vec{x}'(t) = A\vec{x} + \vec{f}(t) \\ \vec{x}(0) = \vec{x}_0 \end{cases}$$

mimic chapter 1!

$$\vec{x}' - A\vec{x} = \vec{f}$$

$$e^{-At} (\vec{x}' - A\vec{x}) = e^{-At} \vec{f}(t)$$

$$(e^{-At} \vec{x})' = e^{-At} \vec{f}$$

$$\int_0^t : e^{-At} \vec{x}(t) - \vec{x}_0 = \int_0^t e^{-As} \vec{f}(s) ds$$

$$e^{-At} \vec{x}(t) = \vec{x}_0 + \int_0^t e^{-As} \vec{f}(s) ds$$

$$(e^{-At})^{-1} = e^{At}! \quad (\text{why??})$$

$$\text{so } \vec{x}(t) = e^{At} \vec{x}_0 + e^{At} \int_0^t e^{-As} \vec{f}(s) ds$$

(is a special case of box on left).

Exercise 2 (p. 367.)

$$\text{Solve } \begin{cases} \vec{x}' = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \vec{x} - \begin{bmatrix} 15 \\ 4 \end{bmatrix} t e^{-2t} \\ \vec{x}(0) = \begin{bmatrix} 7 \\ 3 \end{bmatrix} \end{cases}$$

using $\vec{x}(t) = e^{At} \vec{x}_0 + e^{At} \int_0^t e^{-As} \vec{f}(s) ds$
(and technology help!)

```
> with(LinearAlgebra):  
> A := Matrix(2, 2, [4, 2, 3, -1]);
```

$$A := \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$$

(1)

```
> Eigenvectors(A);
```

$$\begin{bmatrix} 5 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 & -\frac{1}{3} \\ 1 & 1 \end{bmatrix}$$

(2)

$$\lambda = 5$$

$$\vec{v} =$$

$$\lambda = -2$$

$$\vec{v} =$$

So, if you were using method of undetermined coeff's what would be your guess

$$\vec{x}_p(t) =$$

```
> x0 := Vector([7, 3]);
f := t -> t * exp(-2*t) * Vector([15, 4]);
```

$$x0 := \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

$$f := t \mapsto t e^{-2t} \text{Vector}([15, 4]) \quad (3)$$

```
> f(s); #just checking
```

$$\begin{bmatrix} 15 s e^{-2s} \\ 4 s e^{-2s} \end{bmatrix} \quad (4)$$

```
> MatrixExponential(A, -s) f(s); #my integrand
```

$$\begin{bmatrix} 15 \left(\frac{1}{7} e^{2s} + \frac{6}{7} e^{-5s} \right) s e^{-2s} + 4 \left(\frac{2}{7} e^{-5s} - \frac{2}{7} e^{2s} \right) s e^{-2s} \\ 15 \left(\frac{3}{7} e^{-5s} - \frac{3}{7} e^{2s} \right) s e^{-2s} + 4 \left(\frac{6}{7} e^{2s} + \frac{1}{7} e^{-5s} \right) s e^{-2s} \end{bmatrix} \quad (5)$$

```
> integrand := map(simplify, %);
```

$$\text{integrand} := \begin{bmatrix} s(1 + 14 e^{-7s}) \\ s(7 e^{-7s} - 3) \end{bmatrix} \quad (6)$$

```
> int(s(1 + 14 e^{-7s}), s=0..t);
int(s(7 e^{-7s} - 3), s=0..t);
```

$$\frac{2}{7} + \frac{1}{2} t^2 - 2 t e^{-7t} - \frac{2}{7} e^{-7t}$$

$$\frac{1}{7} - t e^{-7t} - \frac{1}{7} e^{-7t} - \frac{3}{2} t^2 \quad (7)$$

```
> x := t -> MatrixExponential(A, t) x0 + MatrixExponential(A, t) . Vector([
- 2 t e^{-7t} - \frac{2}{7} e^{-7t}, \frac{1}{7} - t e^{-7t} - \frac{1}{7} e^{-7t} - \frac{3}{2} t^2]);
```

```
> simplify(x(t));
```

$$\begin{bmatrix} -\frac{1}{7} e^{-2t} + \frac{50}{7} e^{5t} + \frac{1}{2} e^{-2t} t^2 - 2 e^{-2t} t \\ \frac{25}{7} e^{5t} - \frac{4}{7} e^{-2t} - e^{-2t} t - \frac{3}{2} e^{-2t} t^2 \end{bmatrix} \quad (8)$$

← compare to page 368