Math 2280-1
Wed. March 16

§5.5 cont'd.

Recall

for the 1^{st} order system of DE's

$$* \quad \vec{x}'(t) = A\vec{x}$$

- $\Phi(t)$ is a FMS if it's n columns are a basis for the soltn space to $*$ (i.e. $\Phi(t)$ is the Wronskian matrix for a basis to the soltn space)

- $\Phi(t)$ is a FMS iff $\Phi'(t) = A\,\Phi(t)$ and $\Phi(0)$ is invertible.

now continue Tuesday notes, page 3, about all FMS's, and about the special one called e^{At}. $\left(= \Phi(t)\Phi(0)^{-1}\right)$.

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more examples

example 2 $\quad A = \Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & & \ddots \\ 0 & & & \lambda_n \end{bmatrix}$; $\quad \Lambda^2 = \begin{bmatrix} \lambda_1^2 & & 0 \\ & \lambda_2^2 & \\ & & \ddots \\ 0 & & & \lambda_n^2 \end{bmatrix}$; $\quad \Lambda^k = \begin{bmatrix} \lambda_1^k & & 0 \\ & \lambda_2^k & \\ & & \ddots \\ 0 & & & \lambda_n^k \end{bmatrix}$.

so $\quad e^{At} = I + t\Lambda + \frac{t^2}{2!}\Lambda^2 + \dots + \frac{t^k}{k!}\Lambda^k + \dots$

$$= \begin{bmatrix} 1 + t\lambda_1 + \frac{t^2}{2!}\lambda_1^2 + \dots + \frac{t^k}{k!}\lambda_1^k + \dots & & & 0 \\ & 1 + t\lambda_2 + \frac{t^2}{2!}\lambda_2^2 + \dots & & \\ & & \ddots & \\ 0 & & & 1 + t\lambda_n + \frac{t^2}{2!}\lambda_n^2 + \dots \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ & & \ddots \\ 0 & & & e^{\lambda_n t} \end{bmatrix}$$

could you see this another way?

example 3

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = Rot_{-\pi/2}$$

power series

$$A^2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$$

$$A^3 = AA^2 = -A$$

$$A^4 = (A^2)^2 = I$$

so
$$e^{At} = I + tA - \frac{t^2}{2!}I - \frac{t^3}{3!}A + \frac{t^4}{4!}I + \frac{t^5}{5!}A - \dots$$

$$= \cos t\; I + \sin t\; A$$

$$= \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \quad !!$$

could you do this with a FMS approach?

example 4

A is called nilpotent iff $A^k = 0$ for some k. In this case the power series for e^{At} truncates!

e.g.

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad A^2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad A^3 = [\,0\,] \qquad A^k = 0 \;\forall k \geq 3$$

thus
$$e^{At} = I + tA + \frac{t^2}{2!}A^2 = \begin{bmatrix} 1 & t & 2t - \frac{t^2}{2} \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{bmatrix} \qquad !$$

example 4b If $B = \lambda I + A$ where A is nilpotent,

then

$$e^{Bt} = e^{t\lambda I + tA}$$

$$= e^{t\lambda I}\, e^{tA} \qquad \left(e^{C+A} = e^C e^A \text{ if } AC = CA \right)$$

$$= \underbrace{e^{\lambda t} I}_{\text{see example 2}}\, e^{tA}$$

so
$$e^{t\begin{bmatrix} 3 & 1 & 2 \\ 0 & 3 & -1 \\ 0 & 0 & 3 \end{bmatrix}} = e^{3t}\begin{bmatrix} 1 & t & 2t - t^2/2 \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{bmatrix}.$$